THE UNIVERSITY OF ALBERTA

THE MATHEMATIZING MODE

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF EDUCATION

DEPARTMENT OF SECONDARY EDUCATION

EDMONTON, ALBERTA

JULY, 1968

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Mathematizing Mode," submitted by Ross J. Johnston in partial fulfilment of the requirements for the degree of Master of Education.

ACKNOWLEDGEMENTS

The writer wishes to express his appreciation to the Department of Secondary Education for the opportunity and financial assistance afforded him in the preparation of this thesis.

Particular thanks are extended to the Chairman of the thesis committee, Professor S.E. Sigurdson, and the members, Dr. G. Murphy and Dr. K.A. Neufeld for their helpful guidance.

The author also wishes to extend thanks to Professors S.E. Sigurdson and T. Kieren, and also to the other members of the Ed. C.I. 680 class for their unselfish sharing of ideas and their constructive criticism.

And finally, this thesis is dedicated to Judy and Mark, whose inspiration and encouragement made possible its completion.

ABSTRACT

This thesis reports the results of an investigation into the nature of discovery teaching. One mode of discovery, the Mathematizing Mode, is described and applied. It is to be noted from the outset that there will be no attempt to evaluate the Mathematizing Mode or to compare it with other modes of instruction. The purpose of the study is one of description and hence, the results of the study center about the clarity and applicability of this description, and are observational in nature rather than statistical.

A review of the writing of selected authors indicated that there is much confusion as to a precise description of discovery teaching. However, certain elements did appear as common. These common elements, and the theoretical positions held by those authors reviewed, formed the basis for the development of the theoretical framework of the Mathematizing Mode.

The theoretical framework for the Mathematizing Mode was then described as a sequence of four teaching stages. Each stage was further defined in terms of prescribed teacher and pupil behaviors. This framework was then applied to two units of instruction, "The Linear Function," and "The Quadratic Function." During this application, continual use was made of the text, <u>Secondary School Mathematics, Grade</u>

Eleven, by W.B. MacLean *et al*.

The results of this description and application include the following observations:

(1) The Mathematizing Mode is a form of discovery teaching.

(2) The Mathematizing Mode is distinctively described.

(3) Since the theoretical framework of the Mathematizing Mode can be reduced to operational terms, then it can be applied to teach specific mathematical content.

(4) The Mathematizing Mode can be used to develop and teach mathematics at the senior high school level.

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CHAPTER I

THE PROBLEM

I. INTRODUCTION TO THE PROBLEM

That there is a need for research into the process of education is obvious. A look at industry suffices to convince us that advances in technology and methods of mass production are in themselves useless without equal advances in methods of distribution. Drawing a comparison to education, and to mathematics education in particular, we can see that rapid updating is taking place in the development of new mathematical content, and this is being quickly introduced into school curricula. To ensure that these curricula are made available to present and future generations, there must be an equal updating of teaching methods. This analogy illustrates a current dilemma in educational practice. Sufficient emphasis has not been given to research into the process of instruction. Research of this type is necessary in order to devise the best ways to teach new and changing content. The lack of research in instruction is particularly evident at the senior high school level. It is not surprising, in view of the recency of improved curricula at this level of instruction, that researchers have not pursued study in this area. However,

there is no necessity for further delay. In fact, priority should now be given to research directed at updating instructional methods at the senior high school level.

The problem of updating teaching processes in mathematics has not just been recently uncovered. Even a brief examination of the writings of such educational philosophers as Socrates, Whitehead or Dewey reveals serious concern over this issue. It is one that has been the object of great interest since philosophers first directed their attention towards a theory of education.

II. BACKGROUND

Traditionally, educational theorists fall into two broad categories: those who feel that education is a drawing out of something within the learner, and those who view education as a filling up of the mind with knowledge. Educators who adhere to the former philosophy are generally considered to favor a discovery type of teaching process in which the learner uncovers knowledge through personal effort. Those educators who adhere to the latter philosophy are generally considered to favor an expository type of teaching process in which the learner depends upon the teacher to impart knowledge to him. The controversy over these two points of view is an ancient one. In fact,

the derivation of the word education gives us a clue to its antiquity.

> Our word "education" derives either from the Latin word educere or educare. . . . Educere means "to draw out." Educare means "to put in."[1]

Expository teaching has gained a strong foothold as the acceptable method of teaching, but there are some educators who would challenge the effectiveness of this method. The history of education is replete with proponents of a discovery type teaching process. Socrates[2] admits, "I am a sort of gadfly," and Erasmus[3] warns, "children in their earliest stage must be beguiled and not driven to learning."

More recently, Alfred North Whitehead, the British mathematician and philosopher, who spent much of his life in North America writes concerning teaching:

> Let the student experience the joy of discovery now. Ideas which are not utilized are positively harmful. With good discipline, it is always possible to pump into the minds of a class a certain quantity of inert knowledge. You take a textbook and make them learn it. So far, so good. The child then knows how to solve a quadratic equation. But what is the point of teaching a child to solve a quadratic

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[1]Cole S. Brembeck, The Discovery of Teaching (Englewood Cliffs, New Jersey: Prentice Hall, 1962), pp. 59-60.

[2]*Ibid.*, p. 60.

[3]*Ibid.*, p. 61.

> equation? There is a traditional answer to this question. It runs thus: The mind is an instrument, you first sharpen it, and then use it; the acquisition of the power of solving a quadratic equation is part of the process of sharpening the mind. Now there is just enough truth in this answer to have made it live through the ages. But for all its half-truth, it embodies a radical error which bids fair to stifle the genius of the modern world. . . . I have no hesitation in denouncing it as one of the most fatal, erroneous and dangerous conceptions ever introduced into the theory of education. The mind is never passive; it is a perpetual activity, delicate, receptive, responsive to the stimulus. You cannot postpone its life until you have sharpened it. Whatever interest attaches to your subject matter must be evoked here and now; whatever powers you are strengthening in the pupils, must be exercised here and now; whatever possibilities of mental life your teaching should impart must be exhibited here and now. That is the golden rule of education, and a very difficult rule to follow.[4]

Whitehead has isolated one important aspect of learning, i.e. activity on the part of the learner, and contrasted this with the passivity of a knife blade being sharpened.

One of the most important modern treatises on inquiry is the work of John Dewey.[5] According to Dewey, inquiry is the "active persistent and careful consideration of any belief or supposed form of knowledge in the light

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[4]Alfred North Whitehead, The Aims of Education (New York: McMillan Company, 1959) pp. 8-9.

[5]John Dewey, How We Think: A Restatement of the Relation of Reflective Thinking to the Education Process (Boston: D.C. Heath and Company, 1933), p. 9.

of the grounds that support it and the further conclusions to which it tends."[6] Inquiry generally aims at the grounding of belief through the use of reason, evidence, inference and generalization. A person is prompted to engage in reflective inquiry when he faces a forked-road situation or a perplexing problem that causes some discomfort. Thus, thinking moves from a state of doubt or confusion, (the prereflective state), to a situation characterized by satisfaction and mastery over the initial conditions that gave rise to doubt and perplexity, (the post reflective state). In between these two states of mind there are, according to Dewey, five phases of reflective thought which may be characterized as follows: (1) suggestion, (2) intellectualization, (3) hypothesis, (4) reasoning, (5) testing the hypothesis.[7] But as Dewey admits, there is nothing sacred about the foregoing sequence nor is there anything magic about the number five. However, Dewey was interested in presenting the student with a problem for his own resolution.

In spite of these efforts, tradition continues to

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[6] Ibid., p. 9.

[7] Ibid., pp. 106-118.

look upon the teacher as a mere dispenser of knowledge, and the student as a passive receiver. In this tradition, it would seem that the major criterion for being adjudged a good teacher is a firm mastery of the content of a particular discipline, and this criterion apparently becomes more important as the level of education rises. In fact, the noticeable lack of research in instruction at the secondary and post secondary levels lends support to this point of view.

More recently however, curriculum researchers have shown interest in the reorganization of mathematical content and in the introduction of new mathematical content at the various grade levels. Of equal importance, contemporary educationalists are re-examining the methodological approaches of Whitehead and Dewey, and they insist that the teaching of mathematics must include a more active participation on the part of the student than is evident during a conventional expository lesson.

III. RECENT DEVELOPMENTS

It is generally agreed that present trends toward curriculum reform are more than justified. Static mathematical content has tended to become obsolete in the face of the fantastic rate at which the present body of mathematical knowledge is increasing. Apart from this

revision of content, however, the ever present problem of teaching the new materials exists. Simply introducing new content into the curriculum solves nothing, as in all probability the new content will be outdated much more rapidly than the last. In fact, it is hoped that this will be the case, for

> . . . if the matter were to end there, the result might well be disastrous. New curricula would be frozen into the educational system that would come to possess, in time, all the deficiencies of curricula that are now being swept away. And in all likelihood the present enthusiasm for curriculum reform will have long since been spent; the "new" curricula might remain in the system until, like the old, they become not only inadequate but in fact intolerable.[8]

Along with the development of modern curricula and texts, there is an evident trend towards improved teaching methods. The direction of this trend seems to be towards a more discovery-oriented approach to teaching. The following are excerpts from An Analysis of the New Mathematics Programs[9]:

> To accomplish its purpose effectively, the GCMP makes extensive use of both the logical structure of mathematics, and the discovery approach to learning. . . . The GCMP has been guided by the belief that

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[8]Goals for School Mathematics (Boston: Houghton Mifflin Company, 1963), p. viii.

[9]An Analysis of the New Mathematics Programs (Washington, D.C.: The National Council of Teachers of Mathematics, 1963).

> computational skills should be introduced only after the concepts necessary for understanding the particular operations have been developed and the children have demonstrated a grasp of them.[10]
>
> The Madison Project uses an informal, conversational approach to the discovery method. . . . Instead of learning mathematical laws established by others, the pupils formulate their own.[11]
>
> The ideas and materials in the UMaMP are presented in such a manner that the student is led to form the generalization desired.[12]
>
> The SMSG ninth grade material develops most of these topics by using a discovery approach. . . . Throughout the SMSG Geometry all the students are participants. The authors lead them through the intuitive processes that establish a conjecture, and then to formal proof.[13]
>
> The UICSM courses are genuinely concerned with developing precision in the use of the language of mathematics. . . . One of the fundamental concepts of the program seems to be the value attached to the principle of student discovery. Exploration exercises appear frequently, and these are very useful in encouraging and guiding the student in the discovery of generalizations.[14]

With few exceptions, curriculum researchers, in advocating a re-organization of present content, also advocate a change in presentation from the more traditional expository method in which the teacher plays a central role to some type of inquiry method which emphasizes the importance of the learner's position. In view of the great emphasis on improved teaching

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[10] *Ibid.*, p. 11. [11] *Ibid.*, p. 20. [12] *Ibid.*, p. 24.

[13] *Ibid.*, pp. 42-43. [14] *Ibid.*, p. 60.

techniques and the trend towards a more discovery oriented classroom, research in instruction in the area seems to be vital if a common frame of reference is to be established.

IV. STATEMENT OF THE PROBLEM

As has been observed, much research has been carried on in the development of new mathematics curriculum. The results of this research indicate the need for a change in method of presentation from the expository mode to the discovery mode. Unfortunately however, the same zeal exhibited in developing the new curricula has not been exercised in developing a precise description of the discovery mode of instruction which they recommend.

It is the intent of this study to define and develop a distinctive discovery mode of instruction for teaching mathematics--a mode that actively involves the students in the learning situation. The problem for this study, then, breaks down into two phases. The first phase involves a description of a discovery mode of teaching mathematics. Such description must come from considerations of the research in and writing about discovery teaching, and from mathematics and a knowledge of mathematical processes. Based on these considerations, the discovery mode must be further identified in terms of sequence of mathematical content, and of teacher and pupil behavior. And finally,

the dimensions of discovery teaching, as well as their accompanying content sequence and teacher-pupil behavior characteristics, must be incorporated into a well described theoretical framework.

The second phase of the problem involves the appropriate development of curriculum materials which illustrate clearly the implementation of the above description, and their presentation in an actual classroom situation in order to determine the feasibility of their use for teaching present curriculum content.

The difficulty of this problem is intensified by the many meanings of the term discovery teaching, and also by the disagreement among educators as to the primary objective of teaching mathematics.

V. SIGNIFICANCE OF THE STUDY

In view of the problem posed, one question immediately arises. What is the teaching that actively involves the student in the learning situation?[15] It has been called discovery teaching, exploration, hypothesizing. Discovery has been used to describe the Inquiry Method as outlined by Dewey in the early twenties. The Socratic Method of asking leading questions has been labelled by some as discovery teaching. Some studies have equated inductive

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[15]cf. ante p. 9.

learning techniques with the discovery process. This view has been criticized by Shulman's[16] observations that teaching by discovery may not be synonymous with inductive teaching, but rather the discovery process can result from either inductive or deductive teaching. Many modern text books are using such headings as "discovery exercises." The most common general description of discovery is that which first begins with a series of examples from which the students are expected to draw generalizations.

Possibly no other expression is more prominent among educators today than that of "discovery teaching." No other word is tossed about with more recklessness and abandon than "discovery." A need therefore exists for a clearer definition for ease of communication. Davis recognizes this need:

> As is well known, discovery has become a cliché in recent times. It is the darling of many a practitioner (and the <u>bête noire</u> of others), it has been the subject of laboratory studies, and has even been the concern of special meetings (I, Shulman and Keislar, 1966).
>
> Yet there is no agreement on what is meant by either discovery learning or discovery teaching. Nor is there any agreement as to what discovery is supposed to accomplish; hence no evidence of its accomplishing,

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[16]Lee S. Shulman and Evan R. Keislar (eds.), <u>Learning by Discovery: a critical appraisal</u> (Chicago: Rand McNally & Company, 1966), p. 27.

> or not accomplishing, any single objective would change the minds of most of those who do, or do not, believe in it. This is clearly an area where teachers and researchers need more agreement on what they believe they are doing.[17]

This lack of a precise definition also presents a major problem in assessing the value of "discovery vs. expository" experiments.

> One basic difficulty is the lack of precise description of procedures. Studies purportedly designed to assess this question have used the same term to apply to a wide variety of instructional activities. In some experiments the name rote learning is applied to a treatment which other investigations call discovery. Unfortunately, not enough information is supplied to enable the reader to identify the precise procedures under either label.[18]

The agreement on a lack of agreement as to the description of discovery teaching lends significance to this present study. At the same time, before attempting to proceed further, we must clarify the basic assumption underlying the philosophy of this study. It is that present streams of thought in educational practices tend to accept a much broader view of desirable outcomes from a mathematics lesson. A much greater emphasis is being placed on understanding the structure of mathematics rather

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[17]Robert B. Davis, "The Range of Rhetorics, Scale and Other Variables," _Journal of Research and Development in Education_, Vol. 1, No. 1 (Fall, 1967), p. 59.

[18]Shulman and Keislar, _op. cit._, p. 191.

than on discrete learnings. Joseph Schwab,[19] for example, describes one aspect of the structure of a discipline as being the pattern of procedure, the method, how concepts are used and manipulated to attain structural goals. Polya arrives at the same conclusion by stating quite simply:

> Our knowledge about any subject consists of information and of know-how. If you have a genuine bona fide experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is more important than mere possession of information. Therefore, in the high school, as on any other level, we should impart along with a certain amount of information, a certain degree of know-how to the student.[20]

The objectives of a mathematics lesson, then, must be concerned with more than knowledge of mathematics content. Equally important to the learning of content is developing the student's mathematical know-how. The above position regarding objectives is implicitly adopted within the framework of this thesis.

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[19] Joseph J. Schwab, "The Concept of the Structure of a Discipline," The Educational Record, Vol. 43 (July, 1962), p. 203.

[20] George Polya, Mathematical Discovery, Vol. 1 (New York: John Wiley and Sons, 1962), pp. vii-viii.

CHAPTER II

REVIEW OF SELECTED RELATED LITERATURE

In view of the amount of literature directed at discovery teaching, a selection must be made. Two considerations are evoked, the nature of the literature and the recency of the literature. The review is organized around the writings about discovery teaching, and an attempt is made to isolate some of the common characteristics of discovery teaching. Again, since the central problem of the present investigation is one of description, only writers who have a contribution to make in this area will be considered. The numerous experimental studies which deal with the effects of discovery teaching will not be considered unless they make a contribution to the description of the discovery mode.

I. DAVID P. AUSUBEL

In the first article reviewed, Ausubel[1] presents his position on discovery learning. It is a negative one because he feels that learning subject matter is more

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[1]David P. Ausubel, "Learning by Discovery: Rationale and Mystique," Bulletin of National Association of Secondary School Principals, Vol. 45 (December, 1961), pp. 18-58.

important than problem solving as an educational objective. This position is emphasized in the title of the article. By using the term "Rationale", Ausubel implies that there is a logical basis for discovery teaching; however, this rationale applies to only a limited set of objectives.

> At the appropriate time and place, and for carefully designed purposes, learning by discovery has its defensible uses and undoubted advantages.[2]

By using the term "Mystique", Ausubel suggests that unwarranted qualities are attributed to learning by discovery.

> Some of its proponents have elevated it (discovery teaching) into a panacea, making exaggerated claims for its uses and efficacy that go far beyond the evidence as well as far beyond all reason.[3]

Although Ausubel does not define precisely what he means by discovery, he does react in such a way as to reveal some of its characteristics, at least as he views them. Some of these characteristics are:

(1) complete pupil autonomy

(2) the dependence of the method upon actual pupil discoveries

(3) the dependence of the method on exhibited creativity and originality of thinking

(4) the presence of activity programs and project lessons

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[2]*Ibid.*, p. 18 [3]*Ibid.*, p. 18

(5) much emphasis on problem solving

(6) direct and immediate concrete experience as a prerequisite for meaningful understanding

(7) much time spent in gathering and analyzing data as in the scientific method.

The problem in assessing these characteristics stems from Ausubel's rejection of problem solving as a central aim of mathematics education. He very sarcastically refers to Bruner's educational objectives as being grandiose and platitudinous; then he proceeds to the following statement of his objectives:

> . . . all human beings are strongly motivated to learn so that they can better understand themselves, the universe, the human condition and the meaning of life--hence if we are concerned with achieving this particular aim of education, we cannot leave its implementation to problem solving and discovery techniques.[4]

These lofty objectives offered in place of Bruner's indicate clearly his rejection of problem solving as a central educational objective.

In conclusion, this article is particulary relevant in that it questions the role of discovery teaching in the classroom.

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[4]*Ibid.*, p. 36.

One particular study of Ausubel's is relevant to this summary, in that it emphasizes the advantages of preparing a student's cognitive structure to assimilate new information by the use of advance organizers.[5] Two factors are involved in the facilitating influence of advance organizers for the incorporation of new knowledge into the cognitive structure of the learner:

(1) the selective mobilization of the most relevant existing concepts for use as part of the subsuming focus for the new learning task, thereby increasing the tasks familiarity and meaningfulness

(2) the provision of optimal anchorage for the learning material in the form of relevant and appropriate subsuming concepts.

He further adds that the use of advance organizers "would also render unnecessary much of the rote memorization to which students resort because they are required to learn the details of a discipline before having available a sufficient number of key subsuming concepts."[6]

Nowhere is Ausubel's hierarchy of values regarding educational objectives more apparent than in his recent article "Facilitating Meaningful Verbal Learning in the

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[5]David P. Ausubel, "The Use of Advance Organizers in the Learning and Retention of Meaningful Verbal Material," Journal of Educational Psychology, Vol. 51 (October, 1960), pp. 267-272.

[6]*Ibid.*, pp. 271-272.

Classroom."[7] His argument for the effectiveness of meaningful verbal learning (an advanced form of meaningful reception learning) rests entirely on his assumption that the central educational objective is that of learning the subject matter. Three characteristics of reception learning are:

(1) Reception learning is meaningful when the learning task is related to relevant aspects of what the learner knows.

(2) Two factors are involved in the choosing of appropriate content: the nature of the material to be learned and the availability of relevant content in the particular learners cognitive structure.

(3) Use of "subsuming concepts" and "organizers" to aid in the learning task.

The above particular characteristics have been singled out because of their relatedness to ideas expressed by Davis, Dienes, Bruner, Suchman and others.

Ausubel stresses in this article the various dimensions of learning and the existing error in thinking

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[7] David P. Ausubel, "Facilitating Meaningful Verbal Learning in the Classroom," The Arithmetic Teacher, Vol. 15 (February, 1968), pp. 126-132.

that all verbal learning is rote whereas all discovery learning is meaningful. He gives for a basic model the two types of learning to be reception and discovery either one may be rote or meaningful depending upon the means used.

In summary, Ausubel recognizes that there are two distinct types of meaningful learning, discovery and reception. For several reasons, however, the balance between the two should be weighted on the reception side. The basis for this argument is three-fold:

(1) because of time involved, discovery learning is generally unfeasible as a primary means of acquiring large bodies of subject matter knowledge

(2) the actual process of discovery *per se* is never required for the meaningful acquisition of knowledge

(3) the learning of subject matter content is a more central objective than is the development of problem solving ability.

II. MAX BEBERMAN

Beberman has written much on the topic of discovery teaching, and in a more positive vein than Ausubel. Consequently, his concepts regarding the characteristics of a discovery lesson are more apparent. In two articles

Beberman[8] evolves the following characteristics of discovery teaching:

(1) the importance of exploration before effective learning takes place

(2) informality in the lessons, and a lack of insistence upon verbalization of generalizations by the students

(3) the use of graphs when possible, but not to the point of routine application.

(4) no attempt on the part of the teacher to have the students justify initial answers

(5) proper nomenclature to facilitate discussion and the importance of students recognizing this fact

(6) formalizing a concept should merely involve a well stated description of what the student already knows.

It has two advantages:

(a) the student experiences pleasure at seeing his discoveries presented in this fashion

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[8]Max Beberman & Bruce E. Merserve, "An Exploratory Approach to Solving Equations," The Mathematics Teacher, Vol. 49 (January, 1956), pp. 15-18.

[8]Max Beberman & Bruce E. Merserve, "Graphing in Elementary Algebra," The Mathematics Teacher, Vol. 49 (April, 1956), pp. 260-266.

(b) the principle will be referred to and used later, and the student will have to recognize it.

(7) students are encouraged to use a "common sense" method, but always to be looking for short-cuts (formulas)

(8) the discussion of a wide range of problems so that students need to rely on basic concepts.

(9) the use of "frames" as a bridge between the symbolism of arithmetic and the symbolism of algebra

(10) the student is expected to contribute ideas, principles, and rules.

These characteristics are also referred to in An Emerging Program of Secondary School Mathematics.[9] In this book, Beberman devotes several pages to another characteristic:

(11) high standard of precision in the teacher's use of language e.g., explicit recognition of the distinction between symbols and their referents.

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[9]Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge: Harvard University Press, 1958).

This same precision, however, is not necessarily required of the students.

In the preface to the teacher's edition of High School Mathematics-Course I,[10] Beberman is quite explicit in his instructions to the teachers regarding the characteristics of his techniques. He strongly advises that only those teachers who have had sufficient in-service training in the philosophy and method of the UICSM attempt to use his materials.

On further observation, it is quite evident that Beberman places a great deal of emphasis on discovery within a well designated area. Although much pupil autonomy is encouraged, it is kept within a well organized, integrated selection of course content. This seems to be a necessity if his ideas are to be used by a large number of teachers. It is much the same as telling a child he may play anywhere in the house--for purposes of supervision on the part of the parent, and safety on the part of the child.

III. JEROME S. BRUNER

The first article reviewed deals with the effects

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[10]Max Beberman & Herbert E. Vaughan, High School Mathematics Course I (preface to teacher's edition; Boston: D.C. Heath and Company, 1964).

upon children, of active participation in the learning process. Bruner[11] makes an oversimplified distinction between teaching in the expository mode and teaching in the hypothetical mode. In the former, decisions as to mode, pace, style of presentation are principally determined by the teacher; in the latter, the teacher and pupil are more in a co-operative position. Bruner[12] further adds that one cannot describe the teaching process that goes on in either mode with any great precision, but the following two characteristics of discovery teaching are implied: (1) active participation by the pupil in the form of exploration and discussion, (2) students should be armed with an expectancy that patterns exist.

In his book, Towards a Theory of Instruction,[13] Bruner outlines four features:

(1) specification of experiences which most effectively implant in the individual a predisposition towards learning

(2) specification of the method of structuring a body of knowledge so that it can be readily grasped by the learner

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[11]Jerome S. Bruner, "The Act of Discovery," On Knowing - Essays for the Left Hand (Cambridge: Harvard University Press, 1962), pp. 75-96.

[12]Ibid., p. 77.

[13]Jerome S. Bruner, Toward a Theory of Instruction (Cambridge: Belknap Press of Harvard University Press, 1963), pp. 39-72.

(3) specification of the most effective sequences in which to present the material

(4) specification of the nature and pacing of motivation techniques in the process of learning and teaching

In a more recent article, Bruner[14] refers to the problem of teaching so that the child will use effectively what he has learned. He further defines the problem by isolating six subproblems which might be construed as characteristic of teaching by discovery.

(1) The Attitude Problem. The teacher should encourage students to say such things as "Let me stop and think about that"; "Let me use my head."

(2) The Compatibility Problem. The teacher should attempt to bind a new piece of knowledge to an established domain, so that the new knowledge can help retrieve the old. It is really the problem of finding the connection between the learning task and something they already know.

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[14]Jerome S. Bruner, "Some Elements of Discovery," Learning by Discovery - a critical appraisal, edited by Lee S. Shulman and Evan R. Keislar (Chicago: Rand McNally & Company, 1966), pp. 101-114.

(3) Activity Problem. The teacher must afford the opportunity for solving problems so that the student may feel rewarded for the exercise of thinking.

(4) The Skill Problem. The teacher must take time to teach various skills such as:

(a) pushing an idea to its limit

(b) the most efficient way to frame a hypothesis in order that it is testable

(c) conciseness in stating solutions to hypotheses

(5) The Self-Loop Problem. The teacher must accept the student's hypothesis and then turn about and make the student aware of what he has said. Bruner calls this discovery via self consciousness.

(6) The Problem of Contrast. The teacher should approach definition and concepts from the negative direction as well as from the positive direction. Most often a student can define something by what it is not, before he can describe it fully.

This is possibly the best article for actually isolating some of Bruner's ideas on the process of the discovery method.

IV. ROBERT B. DAVIS

"The Madison Project's Approach to a Theory of Instruction"[15] sums up Davis' previous efforts and interest in formulating a theory of instruction.

Davis specifies two kinds of mathematical experiences:

(1) where children do something.

(2) where children discuss something under the leadership of a teacher.

He refers to these as "informal exploratory experiences." These experiences are further defined by listing criteria for choosing them:

(1) Adequate previous readiness must be stressed.

(2) The experience must be related to the fundamental idea.

(3) Student must play an active role.

(4) Interesting patterns should lurk under the surface of each task.

(5) The experiences should be appropriate to the age of the child.

(6) The sequence of informal exploratory experiences must seem to add up to something worthwhile.

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[15]Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction," Journal of Research in Science Teaching, Vol. 2 (June, 1964), pp. 146-162.

Other characteristics of a Madison teacher are also referred to:

(1) Experience lessons are structured as little as possible.

(2) Seminar discussion is somewhat more structured, yet flexibility is maintained.

(3) Teacher-imposed external re-inforcement is not used to determine how questions are answered or how problems are attacked.

(4) Intrinsic rewards such as reduction of cognitive strain or opportunity to inform the class and teachers of recent discoveries are encouraged.

(5) Children should have a method for evaluating an answer which is independent of text and teacher.

(6) With reference to Piaget's theory of gradual modification of existing cognitive structure, two considerations are pertinent.

(a) Readiness building and unstructured exploration are essential for the child so that he may build some basic cognitive structure in which to assimilate new knowledge.

(b) Open-ended lessons are not frowned upon nor is there any great attempt to protect the child from ever conceiving a wrong idea.

Finally, Davis gives us a warning regarding the dangers of short-cutting and the too rapid adoption of formulas to replace common sense methods.

Davis takes a novel approach to the problem of describing discovery teaching in his article, "Discovery in the Teaching of Mathematics"[16] and in the discussion article which follows, "Teaching and Discovery."[17]

> After the showing of his film was completed Davis turned to the participants and asked, "All right gentlemen. You saw the film. What did I do?"[18]

Here are some of the answers given by the participants:

(1) The term which best characterizes Davis' instructional tactics was "commando teaching," genuinely eliciting responses from the students, rather than telling them the answers.

(2) It differs from pure discovery in that the students are not just "messing around."

(3) It is like discovery in that the students are called upon to discover or invent materials to fill in gaps.

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[16]Robert B. Davis, "Discovery on the Teaching of Mathematics," _Learning by Discovery - a critical appraisal_, edited by Lee S. Shulman and Evan R. Keislar (Chicago: Rand McNally & Company, 1966), pp. 114-128.

[17]Lee S. Shulman and Evan R. Keislar (eds.), "Teaching and Discovery" (discussion), _Learning by Discovery - a critical appraisal_ (Chicago: Rand McNally & Company, 1966) pp. 129-132.

[18]_Ibid_., p. 129.

(4) The students, though inventing or discovering constantly, were making the discoveries that Davis wanted.

(5) There are other guides to student exploration than the teacher. A student seeks to check his discovery against reality. The other students also act as a guide to the individual.

(6) Several participants agreed that Davis' method, although good for group teaching would not work for individual teaching.

These comments seem to be in agreement with Davis' original description of his method.

V. ZOLTAN P. DIENES

In two of his books, Building Up Mathematics[19] and the Power of Mathematics,[20] Dienes clearly states his position as a firm advocate of a teaching method oriented towards student discovery.

> The basic skill underlying all scientific and technological skills is control of the tools of

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[19]Zoltan P. Dienes, Building Up Mathematics (London: Hutchinson Educational Limited, 1960).

[20]Zoltan P. Dienes, The Power of Mathematics (London: Hutchinson Educational Limited, 1964).

> mathematical structures, and not enough young people even become aware of their existence.[21]

Dienes recognizes the research of Piaget and Bruner in formulating his description of the learning process.

(1) Dynamic Principle. Preliminary, structured and practice games must be provided as necessary experiences from which mathematical concepts can eventually be built.

(2) Constructivity Principle. Construction should always precede analysis.

(3) Mathematical Variability Principle. Concepts involving variables should be learned by experiences involving the largest possible number of variables.

(4) Perceptual Variability Principle. The same conceptual structure should be presented in the form of as many perceptual equivalents as possible.

Dienes suggests also that learning should take place individually or in small groups of two or three. He further suggests the use of assignment cards arranged both in series (building up a concept by a series of related tasks), and in parallel, (presenting the same conceptual

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[21] Ibid., p. 7.

idea in different material). Something of this type would seem to be a necessity if the teacher had to teach in such small groups.

In his writings, Dienes endorses the work of Beberman, and acknowledges that much learning takes place before the child can verbalize adequately or make precise use of mathematical symbolism. He also insists however, that eventually the time comes when it becomes necessary to resort to symbolism. Between these two stages, Dienes suggests that students develop their own rudimentary type of symbolism.

In a more recent article Dienes[22] outlines further the stages of discovery learning.

(1) A preliminary groping period during which the student explores and manipulates the environment.

(2) Two types of mathematical play are referred to--manipulative and representational. The one follows from the other.

(3) Abstractions, gathering together a number of different events into a class, are formed

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[22]Zoltan P. Dienes, "Some Basic Processes Involved in Mathematical Learning," Research in Mathematics Education (Washington, D.C.: National Council of Teachers of Mathematics, 1967), pp. 21-34.

from a sufficient variety of mathematical materials.

(4) Generalizations, the extension of an abstract class to a wider class of elements possessing the same or similar properties, are formed by looking at a wide number of situations in which the structure is applicable.

(5) The use of symbolism and its interpretation is best accomplished by an intermediate stage in which children have a hand in the process. They can, in this way, see a necessity for its use.

In conclusion, Dienes suggests that a great deal of research, of the naturalistic or observational type, is needed before any final evaluation of his procedures is possible.

VI. GERTRUDE HENDRIX

In her essay, "Learning by Discovery,"[23] Hendrix outlines three procedures referred to as the discovery method. They are the inductive method, the nonverbal awareness method, and the incidental method. In dealing

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[23]Gertrude Hendrix, "Learning by Discovery," _The Mathematics Teacher_, Vol. 54 (May, 1961), pp. 290-299.

with the first two methods, Hendrix maintains that the fallacy of the inductive method lies in its confusion of verbalization of discovery with the advent of the discovery. For her, this separation of discovery from the process of composing sentences to describe them is a major breakthrough in pedagogical theory. There are three major mistakes that teachers make when attempting to teach inductively.

(1) Calling for generalizations before the students have noticed any similarity among the examples used.

(2) Calling for statements of generalizations when the students possess neither the vocabulary nor rules to formulate such.

(3) Confusing generalizations which are discoverable with generalizations that are arbitrary such as definitions and conventions.

In the Incidental Method the teacher sets the stage for many experiments built around a central project. This method is associated with the Progressive Education era. Hendrix suggests that the confusion between this and the present trends towards discovery teaching methods have soured many educators against anything called a discovery method.

Hendrix gives an explanation of the nonverbal awareness method in her study, "A New Clue to Transfer of Training,"[24] and elaborates on it in her article, "Prerequisite to Meaning,"[25] She further states that: "This failure to recognize that awareness of an entity is independent of the existence of a symbol for the entity promotes pedagogy that is not only wasteful, but often harmful."[26]

VII. GEORGE POLYA

In the preface to his book _Mathematical Discovery Volume I_[27], Polya is concerned with conveying to the reader his ideas about problem solving. He justifies this approach by a consideration of the objectives of teaching mathematics as being two fold; teaching mathematical content, and teaching mathematical knowhow. The characteristics of his teaching method, however, are only implicitly suggested. These include:

(1) allowing opportunity for creative work on an appropriate level

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[24] Gertrude Hendrix, "A New Clue to Transfer of Training," _Elementary School Journal_, Volume 48 (December, 1947), pp. 197-208.

[25] Gertrude Hendrix, "Prerequisite to Meaning," _Mathematics Teacher_, Volume 43 (November, 1950), pp. 334-339.

[26] _Ibid_., p. 334.

[27] George Polya, _Mathematical Discovery, Volume I_ (New York: John Wiley & Sons, 1962).

(2) giving students ample opportunity for practice at problem solving

(3) adequate discussion of solutions to problems as presented by the pupils

(4) formation of discussion groups in which one pupil plays the role of teacher

(5) overview of solutions to problems; seeking an easier way, a more concise presentation, a pattern.

In his article "On Learning, Teaching and Learning Teaching,"[28] Polya is much more explicit as to his method of teaching. His foremost objective is to teach young people to think. Polya's ideas are expressed in his three principles of learning:

(1) Active Learning. He advocates that students be allowed to discover for themselves as much as is feasible. The students should actively contribute to the formulation of the problem at hand as well as in its solution.

(2) Best Motivation. The teacher should pay attention to the choice, formulation, and suitable presentation of the problem. The

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[28]George Polya, "On Learning, Teaching and Learning Teaching," American Mathematical Monthly, Volume 12 (June-July, 1963), pp. 605-618.

problem should be of interest to the student, related if possible, to everyday experience.

(3) <u>Consecutive Phases</u>. Polya sees learning as a series of consecutive stages:

(a) exploration of the problem situation

(b) formalization of concepts and patterns

(c) assimilation of these ideas

(d) some time should be reserved for retrospective discussion of finished solutions.

VIII. SUMMARY

In the review of the literature, several characteristics of discovery teaching emerge as common. Most explicit are those referring to the initial exploration period and the use of classroom discussion. In the use of symbolism there seem to be three stages: (1) preverbal intuition, (2) an intermediate stage in which pupils initiate their own symbolism, and (3) the use of precise mathematical symbols.

Some divergence of thought regarding the use of discovery in the classroom involves the answer to the question: "Should discovery be used within the curriculum or should it be used as a supplementary exercise?" For example, Davis is more random in his use of discovery than is Beberman.

Ausubel is included in the review in order to present the major argument against the use of a discovery teaching process. It is his contention that the central aim of mathematics education is that of acquiring a knowledge of subject matter. And since, he is equally convinced that discovery teaching is not suited to attain this end, it should not be used as a primary method of instruction.

The theoretical positions examined in this chapter are now incorporated into the theoretical framework of the present study.

CHAPTER III

THEORETICAL FRAMEWORK

I. THE MATHEMATIZING MODE

One of the major difficulties to date in a discussion of discovery teaching methods is that of communication of ideas among educators. In order to overcome this problem of communication, and to present a sound framework for instruction, the Mathematizing Mode is presented as a sequence of teaching stages and by characteristic teacher behavior to accompany each stage. The more common characteristics of discovery teaching are identified in the previous chapter. Here, they are incorporated into the theoretical framework of the Mathematizing Mode of instruction. As the name indicates however, more than just discovery is involved. The name "Mathematizing" implies doing mathematics; it implies activity on the part of the learner; it implies application of mathematics already discovered; it implies acting as a mathematician.

Whitehead[1] has isolated activity on the part of the student as being one of the most important ingredients of

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[1] Alfred North Whitehead, The Aims of Education (New York: McMillan Company, 1959), pp. 8-9.

a learning situation. More recently Polya, as one of his three principles of learning places "active learning" first:

> It has been said by many people, in many ways that learning should be active, not merely passive or receptive: merely by reading or listening to lectures or looking at moving pictures without adding some action of your own mind, you can hardly learn anything, and certainly you cannot learn much.[2]

Although it is not possible to predict exact pupil behavior during a teaching lesson, implicit in the Mathematizing Mode is the conviction that to learn, the pupil must make an active contribution to the learning process. Thus, teacher behavior is designed to encourage extensive pupil participation through personal inquiry, classroom discussion, and directed applications of mathematical principles.

II. DEFINITION OF TERMS

The Mathematizing Mode. This is a term that has been coined in order to describe precisely one particular discovery type process of instruction. There are many processes that could be classified under the name discovery. It is our intention that by using the phrase Mathematizing Mode, the redundancy of meaning attached to discovery mode will be avoided.

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[2]George Polya, "On Teaching, Learning and Learning Teaching," American Mathematical Monthly, Volume 12 (June-July, 1963), pp. 605-606.

Activity. An activity is any problem situation in which the students are faced with the task of exploring a new mathematical principle, and making hypotheses about the solution.

Teaching Stage. In reference to the Mathematizing Mode of instruction "teaching stage" is a phase of instruction identified in terms of teacher behavior designed to elicit pupil activity in one form or another. There are four such stages in the theoretical framework of the Mathematizing Mode.

Exercise. In the Mathematizing Mode, traditional exercises play a dual role. On the one hand, they may be employed in the traditional sense of practice in the application of a specific mathematical concept. They may also be used as an activity to begin a cycle of the four stages of the Mathematizing Mode. In this sense, exercises may be assigned to be done without the aid of formulas or short-cuts, but from which hypotheses are generated by the students. When used in the latter sense, emphasis is placed on assigning very few problems which are analysed thoroughly in class discussion.

III. STAGE ONE

The first stage on the Mathematizing Mode is characterized by a period of uninhibited exploration of the

problem situation on the part of the pupil. After presenting a suitable problem situation (activity) within a broad framework, the teacher allows time for the students to look for and find existing relationships, patterns and possible solutions. The students are encouraged to examine the problem and formulate plausible hypotheses from which generalizations can arise, or from which further significant problems can be posed.

This phase of instruction is referred to by Davis[3] as "experience without formal instruction" or by Polya[4] as "exploration." Shulman[5] lends great importance to this aspect of learning when he states "No matter what kind of discovery teaching . . . do not lose sight of the importance of the antecedent exploration in the field."

There are two important aspects of the problem situation itself that need elaboration. First, the problem (activity) should be stated within the broadest possible framework. Here we are advocating, to borrow from Piaget's vocabulary, that learning best takes place in terms of

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[3]Robert B. Davis, "Introductory Statement," An Analysis of the New Mathematics Programs (Washington, D.C.: National Council of Teachers of Mathematics, 1964), p. 17.

[4]George Polya, "On Teaching, Learning and Learning Teaching," American Mathematical Monthly, Volume 12 (June-July, 1963), pp. 610-611.

[5]Lee. S. Shulman & Evan R. Keisler (eds.), "The Meaning of Discovery in Learning" Learning by Discovery - a critical appraisal (Chicago: Rand McNally & Company, 1966), p. 29.

"accommodation" primarily and then in terms of "assimilation." By this it is meant that the problem situation, as initially presented, should demand from the pupils an extension of an already known framework to "accommodate" for the new concepts. In this way, his cognitive structure is prepared for the new data, facts and relationships.

Heretofore, much learning consisted in "assimilating" bits and pieces of information before the cognitive structure of the child was adequately prepared. A continual diet of this type of learning tends to cloud the structure of a subject; whereas the approach suggested tends to enhance the structure of a subject.

Schwab offers the same argument in simpler terms:

> The structures of the disciplines are twice important to education. First, they are necessary to teachers and educators: they must be taken into account as we plan curriculum and prepare our teaching materials; otherwise, our plans are likely to miscarry and our materials, to misteach. Second, they are necessary in some part and degree within the curriculum, as elements of what we teach. Otherwise, there will be failure of learning or gross mislearning by our students.
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> . . . .
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> This shift from catalogues to patterns in the disciplines means, in turn, that teaching and learning take on a new dimension. Instead of focusing on one thing or idea at a time, clarifying each and going on to the next, teaching becomes a process of focusing on points of contact and connection among things and ideas, of clarifying the effect of each thing on the others,

> of conveying the way in which connection modifies the participants in the connection. In brief, teaching is the task of portraying phenomena and ideas not as things in themselves, but a fulfillment of a pattern.[6]

Ausubel,[7] who is not noted for his sympathy towards discovery type teaching methods, does suggest that the use of organizers and subsumers which "provide scaffolding for the material to follow, and help to organize the related new facts around a common theme" are important aids to the learning process.

The second aspect of the problem situation which needs explanation is that the problem should be stated in an incomplete form. Ideally, the student should initiate the problem situation (activity), but it is highly impractical to wait for students to arrive at a meaningful problem situation in the initial stages of instruction. However, by stating a problem in an incomplete way, and within a broad framework, opportunity is supplied for the students during the course of their exploration period to redefine and clarify the original problem. In this way the students play an active role in the formulation

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[6]Joseph J. Schwab "The Concept of the Structure of a Discipline," The Educational Record, Volume 43 (July, 1962), pp. 197-202.

[7]David P. Ausubel, "Facilitating Meaningful Verbal Learning in the Classroom," The Arithmetic Teacher, Volume 15 (February, 1968), pp. 130-131.

of the problem.

The objective then of this first stage of the Mathematizing Mode is to place students in a mathematical environment to which they can react with the hope of getting feedback from the subject matter itself. Davis refers to this as coming face to face with the mathematics itself. Student activity in the form of personal inquiry is the essential characteristic of stage one of the Mathematizing Mode.

IV. STAGE TWO

The second stage of the Mathematizing Mode should take the form of a "brainstorming session." The teacher acts only as a moderator and a scribe. Every suggestion and attempt at a hypothesis is accepted and recorded intact, without any hint of evaluation. Each statement by a pupil is to be rewarded by the teacher as contributing to the learning process. The pupils are encouraged to give freely any solution or partial resolution of the problem situation. It is important that the pupils realize that the formulation of a partially correct or even of a wrong hypothesis is as valuable to the development of the lesson as a correct or complete generalization.

According to Whitehead,[8] this section constitutes the major role of the teacher, to assist discovery on the part of the pupil by stimulating the already active minds to make new discoveries on their own. How? By strengthening and exercising these powers already within them.

This idea is in line with Wittrock's conclusion that:

> Many people feel that surely practice at discovering is best treatment to teach children to discover information by themselves. In any event it may be difficult to convince many that one can learn discovery best by some other route other than simply practising the terminal behavior.[9]

Szabo[10] contends that one of the most important phases of a student's mathematics training is the developing of an inquisitive attitude. He also states that students can learn how to search for patterns and that they will be able to use this knowledge and technique in solving new problems.

Richard Suchman describes one method of developing inquiry skills in pupils:

> The procedure used for making children aware of the inquiry process is something we once termed Inquiry Training. (We have been sorry about the word "training" ever since.) We have produced a series of films starting with physics (we now have

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[8]Alfred North Whitehead, The Aims of Education (New York: McMillan Company, 1959), pp. 8-9.

[9]M.C. Wittrock, "The Learning by Discovery Hypothesis," Learning by Discovery - a critical appraisal, edited by Lee Shulman & Evan Keisler (Chicago: Rand McNally & Company, 1966), p. 36.

[10]Steven Szabo, "Some Remarks on Discovery," The Mathematics Teacher, LXI (December, 1967), pp. 839-842.

> economics and biology, also) which are designed as discrepant events. They pose episodes which the children cannot assimilate without accommodating, or at least analyzing the event itself until assimilation is possible. The film is then the focus and offers the initial motivation. Next, we provided the freedom by allowing the children to ask yes-and-no questions to gather their data. These are questions phrased to be answered by "yes" or "no" but the teacher may qualify the answers where necessary. The questions are not attempts to elicit explanations or theory from the teacher but are strictly for data gathering.[11]

Pupils who were trained in this particular inquiry technique tended to ask a greater number of the right kinds of questions, although the calibre of questions did not seem to improve greatly.

During stage two, as well as in stage three of the Mathematizing Mode the students should be encouraged in the use of "intermediate language." A precise mathematical description should not be required from the pupils until the concept involved has been incorporated into their cognitive structure. The students should realize that the "thing" exists by itself. A name is used to identify it, and a formula may be employed to find it quickly. Precision in the use of a language is an aid

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[11]J. Richard Suchman, "The Illinois Studies in Inquiry Training," Journal of Research in Science Teaching, II, No. 3 (September, 1964), pp. 231-232.

to remembering and discussing a useful concept. It should not be made a stumbling block to the learning process. Hendrix[12] describes this "nonverbal awareness" method as one in which a student's learning is lessened to not requiring him to verbalize the generalization being taught. For example, a student may have learned the manipulation of multiplying fractions without being able to express verbally what he has done. Hendrix[13] did a study using the concept, "The sum of the first n odd integers is "n^2." Her results indicated that the students who discovered the concept independently and left it unverbalized exceeded in transfer those who first discovered the concept and then verbalized it. This technique of delaying verbalization is characteristic of Beberman's UICSM project.[14] The Leicestershire Mathematics Project[15] has, from its inception in 1958, emphasized the importance of preverbal mathematical thinking, and has used physical material mathematically structured in

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[12]Gertrude Hendrix, "A New Clue to Transfer of Training," Elementary School Journal (XLVIII) December, 1947), pp. 197-208.

[13]Gertrude Hendrix, "Training by Discovery", The Mathematics Teacher (April, 1962), pp. 2

[14]Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge: Harvard University Press, 1958), p. 27.

[15]Zolton P. Dienes, The Power of Mathematics (London: Hutchinson Educational Limited, 1964), p. 9.

such as way as to make it most probable that such preverbal thinking is initiated. Dienes[16] further suggests that children engaged in manipulating mathematical imagery or structured materials evolve their own kind of rudimentary symbolism. A rigorist attitude on the part of the teacher can, in such cases, retard or altogether stop learning.

In stages two and three of the Mathematizing Mode attention is drawn to this emphasis on the nonverbal aspects of mathematical insights and the possible danger of premature verbalization. Not only is it possible to hinder learning, but, without due care in this respect, rote learning of generalizations may take place in an atmosphere of discovery learning.

There is certainly no implication here that a precise verbalization involving rigorous proof and exact symbolism is not an important aspect of mathematical learning. Indeed this is the case. We only wish to emphasize that verbalization is not a necessary indicator that a student understands a generalization. The student can do much in the way of mathematics without calling on all the precision of mathematical language.

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[16] Ibid., p. 135.

Szabo points this out:

> Patterns are, of course, very important. Recognition of patterns is basic in discovery generalizations at all levels from the most elementary to the most advanced. At the most elementary stages one merely tries to have the students become aware of generalizations. Students who recognize the emerging patterns can give evidence of their awareness by using the generalizations involved. No proof is necessary, nor is it desirable to attempt one. In fact, there is evidence to support the fact that too early verbalization of discovered generalization with mathematical immature children can be damaging to the learner, due mainly to lack of verbal facility. Students should be encouraged to give precise verbalization of generalizations after they have exhibited an awareness of those generalizations. Even later, when proof becomes a natural part of the student's mathematical tool chest, manipulation of general statement according to principles and rules of logic can lead the way to more sophisticated methods for discovery of generalizations.[17]

More, however, will be said about the necessity and the usage of symbolism in stage four.

It is most important during stage two that students be convinced that no progress will be made nor learning take place unless they make a contribution at this point towards the lesson development. Student activity in the form of offering hypotheses is the essential ingredient to this stage of the Mathematizing Mode.

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[17]Steven Szabo, "Some Remarks on Discovery," The Mathematics Teacher, LXI (December, 1967), pp. 839-842.

IV. STAGE THREE

The third stage of the Mathematizing Mode is characterized by a critical appraisal of those hypotheses suggested and recorded during stage two. It is at this time that the evaluation of hypotheses is first encountered. The teacher must initiate questions and lead discussions which will enable the students to test out their hypotheses. It is during this stage of the Mathematizing Mode that the students have time to reinforce those hypotheses that stand up under testing and also to unlearn under their own guidance those hypotheses which they can now evaluate as being wrong or only partially correct. The students at this point should become efficient in disproving hypotheses that are false by the use of a counter example. It is during this stage that students should develop an attitude of suspicion regarding hypotheses or generalizations. This is accomplished when students see for themselves that many of their hypotheses are wrong or only partially correct.

During stage three of the Mathematizing Mode, the teacher reveals certain non-discoverable materials, especially definitions and conventions. An example of these would be the accepted definition of the slope of a line as being

"the change in y divided by the change in x", or the conventional way of describing the position of a point in a plane. Certainly there is nothing wrong with a discussion of the various possibilities for describing the slope of a line or the position of a point in a plane, but eventually the pupils must know the existing convention. It is to be noted that a discussion of these existing possibilities will convince the students of the need for a generally accepted definition or convention.

Once the acceptable hypotheses have been isolated, a certain amount of practice should be provided. In choosing questions for practice, much attention should be given to choosing those problems which extend the student's cognitive structure as well as those which merely reinforce the already established hypotheses. In this way, the practice sessions for the present activity can serve as exploratory sessions for the next activity, and continuity of ideas is maintained. If possible the problem situation for the next exploratory sessions should be initiated by the pupils as a result of their practice period. Polya refers to this as a trick:

> Let me recommend to you here just one little practical trick; let the students actively contribute to the formulation of the problem that they have to solve afterwards. If the students have had a share in proposing the problem they will work at it much more actively afterwards. In fact, in the work of the

> scientist, formulating the problem may be the better part of discovery, the solution often needs less insight and originality than the formulation. Thus, in letting your students have a share in the formulation, you not only motivate them to work harder but you teach them a desirable attitude of mind.[18]

If the students are unable to formulate a suitable problem, then the teacher, as described earlier, must do so in broad terms and incompletely in order to allow for the above mentioned pupil participation.

There are still two important teacher characteristics which must be exhibited during stage three if the spirit of the Mathematizing Mode is to be maintained. The first and more obvious is that all avenues of solution are to be considered. There is no one way to solve a problem; there are many equally good ways. Davis refers to this as listening to the child:

> It may seem superfluous to mention that the teacher needs to listen to the child as carefully as he can. Nonetheless - and this may be symtomatic of the teachers view of the learning process - many teachers do not, and many fine and creative responses of students are rejected as wrong because they do not conform to the _a priori_ expectations of the teacher.
>
> Indeed, listening to the child's suggestions appears to us to be one of the cornerstones upon which new

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[18]George Polya, "On Teaching, Learning and Learning Teaching," _American Mathematical Monthly_, Volume 12 (June-July, 1963), p. 613.

> mathematics is built. Don't require the child to read your mind; don't require him to do it your way --show a simple respect for the child's intellectual analytical, and problem solving autonomy and you will discover that he is a much cleverer child than you had ever imagined:
>
> How can the child improve his own personal cognitive structure when it is only the teacher's cognitive structure that is ever discussed.[19]

This attitude on the part of the teacher has many good effects on the class as a whole. It encourages pupil talk; they know that their answers are considered as important. It allows for student reduction of cognitive strain; after consciencious personal inquiry, the pupil is able to report the results of his findings. Finally, it gives satisfaction to the pupil who enjoys revealing to the rest of the class a novel solution to a problem. Each of these factors is a valuable source of intrinsic motivation for the pupil.

The second characteristic has to do with the effective usage of an open-ended lesson. The philosophy of the Mathematizing Mode is that the teacher should not be too anxious to reveal to the students a complete (if there is such a thing) story of the mathematical principle

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[19]Robert B. Davis, "The Madison Projects Approach to a Theory of Instruction," Journal of Research in Science Teaching, Volume 2 (April, 1964), pp. 146-162.

involved. We are certainly not advocating that the Mathematizing teacher should be classified as that type of teacher referred to by Davis[20] who, seeking a tool for sadistic purposes or status fights with students, makes use of discovery by witholding assistance when the child needs it badly. It is not possible to protect children from gathering wrong (partially correct) ideas; hopefully however, by a succession of approximations, these partially correct ideas will evolve into a more complete understanding of the mathematical concepts.

Davis[21] feels that this view of learning mathematics has many direct implications for the classroom. For one thing, it encourages the use of readiness building and preliminary unstructured exploration in order to allow the child to build some basic relevant cognitive structure which more systematic instruction can seek to modify. Again this picture is consonant with the fact that everything anyone of us knows is wrong or at best, partially correct. It is foolish then to expect our students to learn every-

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[20] Robert B. Davis, "The Range of Rhetorics, Scale and Other Variables," Journal of Research and Development in Education, Volume I, Number 1 (Fall, 1967), p. 61.

[21] Davis, op. cit., 1964, pp. 146-161.

thing correctly from the beginning; never to leave the classroom with a wrong ideas, or an unanswered question.

If we consider Cp to be the present cognitive structure of the child, Davis[22] tests the criteria for its acceptability, regardless of its being an imperfect model of the real situation.

(1) Is this particular cognitive structure, Cp, suitable for the assimilation of the ideas with which we are presently working?

(2) Is this particular cognitive structure, Cp, a suitable one from which we can ultimately arrive at a more sophisticated structure Cp+1?

(3) Are the emotional, social and cognitive aspects of our classroom such that the child will easily move from one cognitive structure, Cp, to various more suitable ones, Cp+1, Cp+2 . . .?

(4) At what point is it desirable for the child to become aware of some of the limitations of a given structure, Cp? When should he develop the new structure Cp+1? What should we, as

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[22] *Ibid.*, p. 153.

> the teacher, do in order to play midwife to
> the birth of structure Cp+1 . . .?

Davis concludes with the remark that the student who graduates from such a program of mathematical experiences should among other desirable attributes, be adept in discarding one cognitive structure and replacing it with a more adequate new one, and he should have the wisdom to know when this is advisable.

It is noted that the above teacher behavior is important during all phases of the Mathematizing Mode of instruction, even though it is more applicable during stage three.

Summarizing briefly, stage three of the Mathematizing Mode offers mathematical experiences to the students by way of a teacher moderated, seminar type of classroom discussion and practice exercises.

VI. STAGE FOUR

The fourth stage of the Mathematizing Mode consists in a summary of the mathematical principles involved in the preceding three stages. The teacher must introduce a precise mathematical description of the ideas already discussed. He must make the students fully aware of present day conventions and language but they must also be made

aware that language may change and conventions are subject to revision. He must convince the students that symbolism used in formulae are arbitrarily chosen and that the formulae themselves are useful as aids to memory and as short-cuts to mathematical calculations. As seen in our previous discussion it is sometimes best to delay this stage of the development until several activities of learning have taken place. However, this stage is necessary and must not be omitted in the overall picture of a learning situation.

Beberman[23], while advocating the advantages of preverbal intuition, warns his readers that precision in the use of mathematical symbolism should be a final outcome of instruction. Dienes further adds:

> Carrying out transformations without the help of symbols very soon leads to an impasse, because we cannot keep in mind, let alone process, the information needed to effect further transformation.[24]

The student should be engaged in verbalizing the mathematical principles involved and in assimilating the information into his now prepared cognitive structure.

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[23]Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge: Harvard University Press, 1958), p. 26.

[24]Zoltan P. Dienes, The Power of Mathematics, Hutchinson Educational Limited (London: 1964), p. 15.

He should be actively engaged in the solution of problems using symbolism and formulas. A word here on the danger of short-cuts is in order. We have already mentioned the dangers of premature verbalization and those of assimilation of facts into an inadequately prepared cognitive structure. The students must come to realize that formulae and symbols are only used to describe concisely something that is already known and understood. They are not an end in themselves.

The introduction of this precision in a mathematics lesson is only possible when the students have already acquired a knowledge and understanding of the principles being described. When it should be introduced is unique to a given classroom situation, and only the individual teacher is in a position to judge the feasibility of its use. Eventually, however, the student must realize the importance of possessing the ability to understand, speak, and read mathematical language. This power enables him to advance through other textbooks on his own and to make more sophisticated use of the discovery process. Szabo[25], makes reference to this use of symbolism:

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[25]Steven Szabo, "Some Remarks on Discovery," The Mathematics Teacher, LX (December, 1967), p. 839.

> . . . when proof becomes a natural part of the student's mathematical tool chest, manipulation of general statements according to principles and rules of logic can lead the way to more sophisticated methods for discovery of generalizations.

It is in this latter case that the use of symbolism has a powerful application. When a certain degree of proficiency is exhibited by the pupil in the use of symbolism, exercises may be assigned as an activity in which the pupils, not having a previous knowledge of the applicable formulas, may actually generate hypotheses to derive suitable formulas. In this sense, the students are generating as it were, their own mathematics. Beberman[26] sums up this close relationship between the discovery process and precision of language:

> These two desiderata--discovery, and precision in language--are closely connected, for new discoveries are easier to make once previous discoveries are crystallized in precise descriptions (it is easier to discover how to solve equations when you know what an equation and a variable are!), and skill in the precise use of language enables a student to give clear expression to his discoveries.

VII. SUMMARY

The Mathematizing Mode of instruction consists

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[26]Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge: Harvard University Press, 1958), p. 4.

of the cycling of these four stages. It is suggested that the first three stages go through several cycles before stage four is invoked. It is also suggested that stage one is more loosely structured than stage four and that the stages of instruction become more restrictive as the unit progresses. In fact, the final time stage one occurs in a unit of instruction, it will be progressively more structured than the previous stage ones. This is due to the successive changes in the cognitive structures of the pupils. Also, it is noted that the students may initiate the activities of stage one from their engagement in the discussion and practice sessions of stage three. Ideally, the teacher steps out of the learning situation. In this case the students would be creating mathematics on their own; they would, in the term of Bruner, be acting like mathematicians; or in Davis' words, the students would be brought into a direct face-to-face confrontation with the mathematics itself. This is indeed a desirable outcome for our students. The following table is offered by way of a summary of the four stages of instruction, the teacher behavior accompanying each stage, and the type of activity in which the students should be engaged.

STAGES	TEACHER BEHAVIOUR	STUDENT ACTIVITY
Stage One -		
"Introduction of the Activity"	Presentation of activity (when necessary)	Personal inquiry
	Arm the students with an expectancy that patterns exist.	Search for patterns or short-cuts
Stage Two -		
"Brainstorming Session"	Recording of Hypotheses	Presentation of ideas
	Stress non-verbal aspects of mathematical generalizations	Use of "intermediate language"
		Exhibit inquisitive attitudes
Stage Three -		
"Seminar Type Discussion"	Initiate questions to assist in evaluation of hypotheses (when necessary)	Discussion & Evaluation of hypotheses
		Use of counter-example in disproving wrong hypotheses
	Introduction of necessary conventions & definitions	Exhibit suspicious attitude toward generalizations & short cuts.
	Afford practice exercises which apply and extend the generalizations	Possible suggestions of activities for next stage one
	Pursue many solutions if offered by the students	

STAGES	TEACHER BEHAVIOUR	STUDENT ACTIVITY
Stage Four -		
"Summary"	Definition of mathematical principles in precise language.	Verbalization of mathematical principles.
	Afford practice in the use of symbolism and formulas.	Demonstrate an ability to apply mathematical principles to selected problems.

CHAPTER IV

THE LINEAR FUNCTION

In this chapter, an attempt is made to illustrate the use of the theoretical framework of the Mathematizing Mode. A unit of instruction, encompassing the linear function and related topics is developed, and this is presented to a grade eleven class. It is a test situation to assess the feasibility of using the Mathematizing Mode to present an entire unit of the grade eleven curriculum. In the development and presentation of this unit, an attempt is also made to identify the four stages of instruction as outlined in the theoretical framework. Since two observers were present during the teaching of these lessons, their comments are also included in the description of this unit.

The approach taken to the linear function was that of the cartesian co-ordinate graph. Certainly the graph of a linear function contains as much information or more than either the rule or the table of values, as both are obtainable from the graph. The graph has the added advantage of its visual form. Certain activities are now described which help the student establish a framework for the problem at hand.

I. ACTIVITY ONE

Presentation of the Problem. The students were asked to establish a name for the graph of a straight line in a plane. (The line was presented by means of the overhead projector.)

Comment. The students were allowed to consider the problem for a short time, but it was obvious that other problems were occupying the student's minds. These problems arose through discussion and it was soon established that there was need of some reference point from which to start and also a pair of reference axis through this point.

From here it became apparent that the reference point, (the origin), and the reference axis, (the co-ordinate axis), must have names. It was also established that some type of scale along the axis was needed.

The naming of particular points in the plane was hypothesized, and the students suggested two methods. It is interesting to note that when using ordinary language, "two over and three up" results in the same point as "three up and two over." When ordered pairs of numbers are used however, there is a great difference between (2,3) and (3,2). Herein lies the source of the two accepted methods. At this point the idea of a "convention" was introduced by the teacher, but those students who wanted to use the other

method were not discouraged. For a while then, the class was faced with the problem of giving each point two names. In time however, all the students adopted the conventional method for naming of points in a plane. A prepared overlay of a co-ordinate graph was then superimposed on the line and the problem was reduced to finding a name for the line in the co-ordinate plane in terms of the co-ordinates of the points on the line. In all, it was a good example of a problem stated inexactly by the teacher, and refined by the pupils. It also illustrated a good method for introducing a mathematical convention.

Hypothesizing. The students attacked the problem individually, by finding and predicting more points on the line, and by finding a relation between the co-ordinates of these points which distinguished them from other points on the plane.

Other points on the graph of the line were easily predicted after the general pattern "two up, one across," was observed and also, after working at it for a few minutes several students offered such names for the line as:

(1) $2x + y = 2$

(2) $2x + y - 2 = 0$

Evaluation. A means of checking these rules was applied by the students. Several points on the line were substituted in the equation to see if they yielded true sentences.

Summing Up. There are two conditions which must be satisfied relating to the graph and the rule.

(1) every point on the graph of the line must have co-ordinates that satisfy the rule.

(2) every point whose co-ordinates satisfy the rule must be on the graph.

Comment. Activity one is a rather trival activity for a grade eleven class and many problems were raised and solved through class discussion, which might otherwise have resulted in much personal inquiry and hypothesizing. This also explains why no wrong hypotheses were generated and the brief evaluation stage before summing up the activity.

II. ACTIVITY TWO

Presentation of the Problem. A page of graph paper was distributed to each student, and they were asked to fill in the names for each point, to make any pertinent observations, and to pick out some straight lines of their own and find rules for them.

Comment. After a short while several generalizations were accepted from the students and the teacher wrote these

on the board. Set builder notation was encouraged from the beginning, but other verbalizations, no matter how clumsy, were accepted.

It is important at this point that the teacher must bring himself to appreciate a wrong or incomplete answer, and work with it, as it is just as important to the development of the whole as a correct answer.

Hypothesizing. Here is a partial list of generalizations that were presented by the pupils.

(1) In the upper left corner, x is negative and y is positive.

(2) In the upper right corner, x is positive and y is positive.

(3) In the lower left corner, x is negative and y is negative.

(4) Along the X-axis, $y = 0$.

(5) Along the Y-axis, $x = 0$.

(6) The origin (0,0) is special.

(7) Along any horizontal line, y has the same value.

(8) Along any vertical line, the x value is the same.

(9) Sums of the co-ordinates on a horizontal or vertical line are consecutive integers.

(10) Sum of the co-ordinates along a diagonal line is always the same.

Comment. A discussion of the word diagonal turned out to be quite messy. It was agreed to call a diagonal of a square, "special diagonal," and the diagonal of a rectangle, "ordinary diagonal." This intermediate terminology was soon forgotten, but it is a good example of staying away from precise terminology initially.

Evaluation. Each of the above hypotheses were discussed by the class, and evaluated. One student challenged the last hypothesis, which did indeed prove fruitful. The counter example was the special diagonal through the origin with slope one. Here it was observed quickly that the difference between the co-ordinates of any point was zero. Two families of lines were then identified.

(1) $\{(x,y) \mid x + y = k,\ x,y \in R\}$

(2) $\{(x,y) \mid x - y = k,\ x,y \in R\}$

Comment. It should be noted that the teacher wrote down this convention, set builder notation, arbitrarily.

One student rejected the cumbersome naming, but another insisted that the designation of the replacement sets for x and y was important, or else you would only get a series of points and not a complete line. However, this precise manner of naming the replacement set was soon dropped.

Another significant observation that came up in the discussion with regard to the line $x + y = 0$ was that all points below (to the left) had co-ordinates whose sum was negative, while all points above (to the right) had co-ordinates whose sum was positive. Even simple statements like the one above the teacher tried to put into the form of hypotheses which students were encouraged to accept or reject.

III. ACTIVITY THREE

Statement of the Problem. The students were again supplied with graph paper. On one graph they were asked to draw and name several lines of the form $x + y = k$; on another they were asked to draw and name several lines of the form $x - y = k$; and on a third graph paper draw a line through any three points (an ordinary diagonal) and find the rule for their line.

Comment. The first two parts to the problem directly resulted from the stage three of the previous activity and served as a review. These two were dealt with quickly by the teacher and the hypotheses suggested are related to the third part of this activity.

Several students found names for lines of their own choosing and they hypothesized these rules for the teacher.

The teacher was careful to accept a rule in any form, but he re-arranged it in several ways after it was presented by the pupils.

Hypothesizing. Format for receiving answers from the pupils was as follows:

Points Chosen	Ax+by+C = 0	$y = -\frac{A}{B}x - \frac{C}{B}$
(3,0) , (6,2) , (9,4)	2x - 3y - 6 = 0	y = 2/3x - 2
(0,-4) , (1,0) , (2,4)	4x - y - 4 = 0	y = 4x - 4
(0,0) , (2,3) , (4,6)	3x - 2y = 0	y = 3/2x
(3,0) , (3,1) , (3,2)	x = 3 (Pointed out that this was not a diagonal)	

Evaluation. The students checked the rules after a sufficient number had been accepted from the class. They approved or disapproved on the basis of finding a counter example of a point on the graph which did not satisfy the rule.

Hypothesizing. The most important outcomes from a discussion of these lines (and others) were the ideas of steepness of a line and direction of a line. It was hypothesized by the class that:

(1) All lines running from the lower right corner to the upper left have same signs for the x & y.

(2) All lines running from the lower left to the upper right have different signs for the x and y.

(3) Lines that are parallel have the same slope. (Note that the word slope had not been introduced as yet.)

(4) It is not true that lines which have the same slope are parallel of necessity.

(5) Two points are sufficient to determine the position and direction of a line.

(6) One point and a knowledge of "how many over and how many up" to the next point are sufficient to determine the position and direction of a line.

Evaluation. It was at this point that a discussion of a slope became important to the pupils. There was much discussion about numbers (4) and (5) above. Upon observation of their two summaries of lines of the form $x + y = k$ and $x - y = k$, it was noted that in each case you move one over and one up to predict future points, hence the lines have the same slope. You can see the reasonableness then that, in the eyes of the students, hypothesis (4) is true, but its converse doesn't have to be true. To get around this, the teachers and students agreed to call a slope positive if the line was of the form $x - y = k$ and negative if the line was of the form $x + y = k$.

Comment. The teacher then proceeded to inform the pupils that the slope of a line is given by a ratio of the change in y over the change in x. This led to the

development of a formula for finding the slope of a line, viz.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

and the students had little trouble in applying this to find the slopes of various lines. They were also quick to observe that it was indeed the case that lines from lower left to upper right had positive slope, whereas lines from lower right to upper left had negative slopes. This is an example of a non-discoverable definition being presented to the class after they realized that it was an important thing to know.

IV. ACTIVITY FOUR

Statement of the Problem. The students were given several rules and asked to find the graph of such. They were asked to make hypotheses from the rule, rather than the graph as to methods of finding slopes, intercepts, and relationships between the slopes of parallel and perpendicular lines.

We used as examples some of the rules already suggested from the previous activity as well as others of the following types:

(1) $2x = 3y - 4 = 0$

(2) $3x - 2y + 4 = 0$

(3) $3x + 4y + 7 = 0$

(4) $4x - 3y + 12 = 0$

Hypothesizing. From these and other rules the students generalized to the following hypotheses:

(1) Lines that have the same slopes are parallel.

(2) The slopes between different pairs of points on the same straight line are equal.

(3) Lines that have negative slopes are perpendicular to each other.

(4) The slope of a line as found from the rule is the coefficient of x divided by the coefficient of y.

(5) The x and y intercepts are found from the rule to be respectively $-C/A$ and $-C/B$.

(6) The slope of a line as found from the rule is the negative of the coefficient of x divided by the coefficient of y.

(7) The slopes of perpendicular lines are related such that their product is negative one.

Evaluation. It was found during discussion that hypothesis (3) was the direct result of observing the two lines of the form $x+y=k$ and $x-y=k$.

Comment. A suggestion at this point is that in choosing examples for this type of exercise that rules with numeric coefficients one and zero be avoided as it is

easier to generalize <u>to</u> these forms than <u>from</u> these forms.

<u>Evaluation</u>. Hypothesis (4) was seen to be wrong and all of the students were quite willing to reject it in favor of hypothesis (6).

At this point the teacher introduced the convention of (x,y) being a general point on the line and (x_1,y_1), (x_2,y_2), (x_3,y_3) etc., being specific points.

<u>Presentation of the Problem</u>. As a second part of the same activity the students were supplied with a slope and a point on the line. They were asked then to find the rule and draw its graph.

eg. Find the equation and plot the graph of the line with slope 2/3 and on the point (4,3).

<u>Hypothesizing</u>. A few of the students attacked the problem in the conventional manner:

Let (x,y) be any point on the line

Then the slope of the line is $\dfrac{y - 3}{x - 4}$

But this is equal to 2/3

$$\frac{y - 3}{x - 4} = \frac{2}{3} \qquad 2x - 8 = 3y - 9$$

and the required rule is:

$$2x - 3y + 1 = 0$$

Other students (most of the class in fact,) approached the problem in this way:

Since the slope is given as 2/3, then the rule is

$$2x-3y+C = 0.$$

Since (4,3) is a point in the line, then

$$2(4) - 3(3) + C = 0$$
$$8 - 9 + C = 0$$
$$C = 1$$

And the required rule is:

$$2x - 3y + 1 = 0$$

Comment. Both procedures were accepted by the teacher but further discussion resulted in most of the class adopting the second method. This posed a problem in that the class did not see any necessity for arriving at the following forms of the straight line.

(1) $y - y_1 = m(x - x_1)$ slope-point form.

(2) $y = mx + b$ slope-y intercept form.

(3) $y = m(x - a)$ slope-x intercept form.

(4) $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ two point form.

In fact, when they were next asked to find a rule and draw the graph of a line on two given points, many students recognized immediately that knowing two points was equivalent to knowing the slope and one point. Questions of this type turned out to be just an exercise in the slope-point form, and most used the second method. The students did not become familiar with these various forms of finding equations. In fact, it would be an interesting exercise to

devise an activity which would result in the students feeling a need to develop some or all of these specific short forms. The activity does illustrate however that several methods of solutions should be accepted.

V. ACTIVITY FIVE

Statement of the Problem. Find the distance between two points, and find the co-ordinates of the midpoint of a line segment joining two points. Both of these problems were presented to the pupils and they were asked to find a general way of solving the above two problems.

Hypothesizing. Two suggestions were offered:

(1) use a ruler

(2) use the Pythogorean Theorem

After a discussion about the relationship between inches and units, the question was changed to read, "Find the distance, in units, between any two points."

It was hypothesized that we should use the Pythagorean relationship by first finding the slope associated with the line joining the two points.

One student further hypothesized that if the line joining the two points is parallel to either the X-axis or Y-axis you can't use the Pythagorean Theorem. Upon examination, he reworded it, "You don't need to use the Pythagorean Theorem."

Evaluation. The discussion of whether to use the slope or the exact units resulted in a conclusion that the slope could be used as long as it was not reduced, which was in fact using the exact distances.

Summing Up. The distance formulas was generalized to the form:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Comment. There was more discussion here as to the difference in meaning between

$$\sqrt{a + b} \text{ and } \sqrt{a} + \sqrt{b}$$

The class was also quite concerned over the use of $x_2 - x_1$ or $x_1 - x_2$. Did you always have to use the positive result? Also, as in the finding of slope, did you have to be consistent in using (x_1, y_1) and (x_2, y_2)? Further discussion and some simple examples solved these questions and the students were given a few exercises in which they applied this formula.

Statement of the Problem. The second part of the fifth activity had to do with finding the co-ordinates of the midpoint of a line segment joining two points. They were given the problem of finding the midpoint of the line segment joining (1,3) to (7,5).

Hypothesizing. Two suggestions were:

(1) use similar triangles

(2) take half the distance along the vertical and half the distance along the horizontal.

Evaluation. On pursuing the second suggestion, it was observed that the correct co-ordinates of the midpoint are (4,4) and the method worked.

Summing Up. From here the students offered a generalization:

$$X = \frac{x_1 + x_2}{2} \qquad \text{and} \qquad Y = \frac{y_1 + y_2}{2}$$

where (X,Y) is the midpoint of the line segment joining (x_1,y_1) and (x_2,y_2).

VI. SUMMARY

Chapter four shows the way in which the description of the Mathematizing Mode may be used to develop and teach a specific unit of instruction. The topics covered are those that generally comprise the unit on the linear function and related topics at the grade eleven level. The treatment of the topic follows the general outline of the four stages described in the theoretical framework. Two of the strategies used which might otherwise be overlooked are:

(1) The structuring of stage one becomes more pronounced in the activities towards the end of the unit.

(2) The use of exercises in stage one. This enables a student to make use of the symbolism he has already acquired to generate new hypotheses relating to the solution of these exercises.

Again, it is to be noted that one of the major difficulties is that of keeping a running account of the hypotheses offered by the students. The teacher should devise his own method of doing so. In this particular study the overhead projector was of invaluable assistance.

A suggested improvement for this sequence of activities would be to include one which illustrates the necessity of knowing the various forms for the equation of a straight line. Although each form of the straight line equation is a special case of the slope-point form, there are advantages to knowing how to apply each under special circumstances.

CHAPTER V

THE QUADRATIC FUNCTION

The development of the quadratic function and related topics as well as their class presentation is described in this chapter. A different approach to reporting the quadratic unit is taken from that of the previous chapter. Observer comments are incorporated into the development and presentation of the lessons, and the stages of instruction are not identified. This seems to be a good approach in that the stages of instruction should be obvious to the reader without identification by the author. Students' names are use at times in order to present a more readable description.

Again the topic was introduced through the medium of the graph of the quadratic function. The graph gives the broadest frame of reference to the quadratic function, which in turn allows the teacher and pupils to "zero in" on the quadratic equation.

I. ACTIVITY ONE

The graph of the parabola, $y = x^2$ was displayed on the overhead projector. The parabola was clearly on the points (0,0), (1,1), (2,4), (-2,4), (3,8), (-3,9), and the students were asked to come up with a rule relating

the co-ordinates of the points.

The students were allowed sufficient time to formulate their ideas and the following suggestions were offered:

(1) The space between any two points parallel to the X-axis is the same distance from the Y-axis, and they have the same height.

(2) $x^2 - y = 0$

(3) $y = x$

(4) Other points can be predicted by the rule: "up by two's i.e. 1,3,5,7, etc. along the Y-axis, and over one each time along the X-axis."

(5) The y value for any point on the graph is never smaller than 0.

A discussion of each hypothesis followed, and the class agreed upon the rule;

$$\{(x,y) \mid y = x^2, x \in R, y \geq 0\}$$

The statement of the above rule led to a discussion concerning the domain and the range of $y = x^2$. The discussion concerning the range involved much in the way of leading questions on the part of the instructor, and this procedure was criticized in that it was thought $y \geq 0$ should have been a final learning rather than an initial condition.

The activity was short-lived however, possibly because of its close resemblance to the first activity in the linear section. The importance of the activity was not that the students arrived at the rule $y = x^2$, but rather that the students were allowed to make all sorts of suggestions and really grasp the ideas of the graph. The idea of symmetry was important as well as the idea of a minimum value for y; even the idea of one over, one up; two over, four up; three over, nine up, etc. proved to be beneficial.

II. ACTIVITY TWO

The second activity was to find the effect on the rule of moving the original graph, (same shape as $y = x^2$), to various points in the co-ordinate plane. The students were asked to try to generalize the formation of new rules when the graph is moved in a certain fashion consistently.[1]

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[1]The instructor gave each student a piece of onionskin paper and a piece of graph paper. A graph of the parabola $y = x^2$ was drawn on the onionskin as related to the graph paper underneath. The students were told that they could move the vertex of the graph to any new point provided that they kept the graph in the vertical or horizontal position. They were able to see through the onionskin and observe points on the graph in the new position.

Other suggested approaches to this activity are to supply a cut-out of a parabola and move it around, or to supply a plastic overlay with the graph $y = x^2$ drawn on it. The instructor made good use of the overhead projector during this activity.

They moved the graph around and made observations as to new points and formulated new rules.

Some of the hypotheses resulting from this exploratory period were as follows:

(1) From the origin, if you move the graph up on the Y-axis, the equation changes from $y = x^2$ to $y = x^2 + 1$, $y = x^2 + 2$, etc.

(2) If you move the graph down the Y-axis, the equation changes from $y = x^2$ to $y = x^2 - 1$, $y = x^2 - 2$, etc.

(3) From origin, if you move the graph up or down, (change in y), you add that change to the x^2.

(4) From the origin, if you move it to right or left, (change in x), you add or subtract that change to y.

(5) If you rotate 90° you set $y^2 = x$, (and the tip of the graph doesn't leave the origin.)

(6) If you rotate 180° you set $y = -x^2$.

(7) If you rotate 270° you set $-y^2 = x$.

(8) If you move it over one space on the X-axis you get $x^2 - 2x = y - 1$.

(9) If you move n spaces to the right on the X-axis keeping $y = 0$, you get $y = (x - n)^2$.

(10) If you move n spaces to the left on the X-axis you get $y = (x + n)^2$.

The period was almost at an end, and it was decided to allow the students to use the rest of the time to work on their own to test out these hypotheses.

The first thing, next period, was to put these hypotheses on the overhead projector and to check each one through discussion.

It was found that (8), (9), (10) are really the same. Hypothesis (4) was exemplified by $y + 1 = x^2$ to represent the graph with vertex at (1,0). The rule worked for the point (1,0), but Brenda said that this is only one point. When further points were tried, the generalization was shown to be wrong. Several good questions arose from the discussion.

(1) How do you make a graph wider or narrower?

(2) How do you move a graph up or down and sideways at the same time?

(3) Do these rules work when you invert the graph, and move it around?

A tendency for students to say "But this only works in a special case, how do you do it when etc.?" was quite apparent during the evaluation stage. This tendency on the part of the student to want to generalize to all encompassing situations is very helpful and is indeed a good motivation. It appears that this is exactly what mathematics is about: getting relationships of general types. The student is

then motivated to get general formulas; he wants formulas. If the instructor can bring him one step further along, to make him want to get his own formulas, then indeed he has achieved a major purpose of mathematics education.

III. ACTIVITY THREE

For the third activity, the instructor chose to pose the student initiated problem, "How do you make a graph wider or narrower?" Two graphs, (not the rules), were presented to the pupils on the overhead projector.

(1) $y = \frac{1}{2}x^2$

(2) $y = 2x^2$

The first graph was clearly on the points (0,0), $(1,\frac{1}{2})$, (2,2), $(3,4\frac{1}{2})$, $(-1,\frac{1}{2})$, (-2,2), $(-3,4\frac{1}{2})$.

The second graph was clearly on the points (0,0), (1,2), (2,8), (-1,2), (-2,8).

The students were asked to find a rule for each of these graphs, and after some time spent in personal inquiry, they offered the following suggestions:

For graph (1) $2y = x^2$ or $y = \frac{1}{2}x^2$

For graph (2) $\frac{1}{2}y = x^2$ or $y = 2x^2$

Further practice at testing hypotheses was hindered because no wrong conjectures were offered by the students. This is a typical example where the immediate response of

only correct hypotheses actually impeded the development of the lesson.

The graphs $y = 2x^2$, $y = x^2$, $y = \frac{1}{2}x^2$ were then superimposed one upon the other and projected on the screen by means of the overhead.

(1) $y = \frac{1}{2}x^2$

(2) $y = x^2$

(3) $y = 2x^2$

An observation of these graphs confirmed the pupils' intuitive feeling about the effect of the coefficient of x^2 upon the shape of the graph--as the coefficient of x^2 becomes larger, the graph becomes narrower.

As a sub-activity, the students were asked to move these graphs, $y = \frac{1}{2}x^2$ and $y = 2x^2$, around, and see what relationships held. The following generalities resulted from the discussion:

(1) The general method of moving the graph $y = x^2$ about, has the effect of changing the rule
$y = x^2$ to $y = (x - \Delta x)^2 + \Delta y$

(2) This same general rule applies to moving the graphs $y = \frac{1}{2}x^2$ and $y = 2x^2$.
$y = \frac{1}{2}x^2$ becomes $y = \frac{1}{2}(x - \Delta x)^2 + \Delta y$
$y = 2x^2$ becomes $y = 2(x - \Delta x)^2 + \Delta y$

(3) For $y = x^2$ as x goes up or down 1, y goes up 1

as x goes up or down 2, y goes up 4

as x goes up or down 3, y goes up 9

etc.

(4) For $y = 2x^2$ as x goes up or down 1, y goes up 2

as x goes up or down 2, y goes up 8

as x goes up or down 3, y goes up 18

(5) For $y = \frac{1}{2}x^2$ as x goes up or down 1, y goes up $\frac{1}{2}$

as x goes up or down 2, y goes up 2

as x goes up or down 2, y goes up $4\frac{1}{2}$

(6) For a wider graph, muliply x^2 by some number less than one.

(7) For a narrower graph, multiply x^2 by some number greater than one.

It is clear from the above discussion that the students had a fair understanding of a quadratic function, and the role of the coefficients in determining the shape of the graph. They also had a good working knowledge of the "vertex form" of a quadratic function. Significantly, to date there has been no reference to the general form of the quadratic function.

IV. ACTIVITY FOUR

The fourth activity was designed to get at an understanding of the standard form of the quadratic function and once again the graph technique was used. At this point the

instructor chose to structure the activity considerably by presenting the students with the following series of functions to be graphed:

(1) $y = x^2 + 2x + 1$	(5) $y = x^2 - 6x - 3$
(2) $y = x^2 + 2x + 2$	(6) $y = x^2 - 7x + 2$
(3) $y = x^2 + 4x + 4$	(7) $y = 2x^2 + 4x + 6$
(4) $y = x^2 + 4x - 5$	(8) $y = 2x^2 - 6x - 9$

During the exploratory period that followed, there was considerable class interaction.

Melvin wanted to just plot a few points and fill them in to complete the graph.

Mike didn't agree because "sometimes you can't find the vertex exactly," and besides, "there must be an easier way to do it."

Sharon became quite excited because she knew how to predict other points if she only knew where the "tip" was. "Tell me how do you find the tip!"

Melvin didn't think this was too significant, "You don't have to find the tip because that is just the point where y is smallest."

Keith suggested a formula existed, "Why don't you use the formula for finding when y is smallest?"

Bob said the formula was $-x = \frac{b}{2a}$.

Sharon, who was quite anxious to get at the vertex, wanted to know if it always worked. The instructor replied that he wasn't sure if it worked at all. Sharon was now quite angry, but Bob assured her that it would work.

During the graphing of these functions, several students in addition to solving the original problem, developed on their own the technique of "completing the square." They also became increasingly aware that the "vertex form" of the quadratic function was decidedly more useful than the "standard form."

The instructor then held a discussion on how to find the "completed square" form from the "standard" form. The class ended up with some rules, but nothing definite. Basically, the students just guessed, and then tried to work it out. It was clear however that the problem was that of verbalizing because they did have some system. Here it might have been a wise move to impress upon the students the necessity of becoming skillful in the technique of "completing the square."

The remainder of the period was then used to allow the students to put their work on the board. The variety of methods employed illustrated all of the techniques for graphing mentioned on the previous page. They are:

(1) plotting individual points.

(2) changing to vertex form

(3) making use of the sequence for plotting points 1 over 1 up, etc.

(4) use of the formula to find the x-co-ordinate of vertex.

It is important here to emphasize that the students need to recognize the various methods used and to become skillful in using various techniques.

V. ACTIVITY FIVE

The fifth activity was aimed at arriving at a method for finding the vertex, hopefully a formula, and also at the significance of the negative co-efficient of x^2. This activity is really a rephrasing of the previous activity, but it did, however, present a little variety in dealing with this problem. The problem posed was that of finding the highest (or lowest) point on the graphs of the following quadratic functions.

(1) $y = (x + 2)^2 - 7$	(5) $y = x^2 - 8x + 27$
(2) $y = -x^2 + 3$	(6) $y = -x^2 - 12x - 3$
(3) $y - -2x^2 + 9$	(7) $7 = 2x^2 - 8x + 112$
(4) $y = (x - 2)^2 - 7$	

The students were given time to work on the above exercises. They had no problem applying the results of the first activity in knowing that the graphs of (1), (4), (5), (7) opened upwards and y had a smallest value,

whereas the graphs of (2), (3), (6) opened downwards and y had a highest value. From the discussion which followed, these observations resulted:

(1) The students determined that $(\quad)^2$ was always positive.

(2) The highest or lowest value of y would be when the $(\quad)^2$ term was zero.

(3) The value of y at this point was just the constant term.

Bob wanted to solve this problem of finding the highest or lowest point on the graph by virtue of the knowledge that the vertex was this point and just using the old format $y = (x - \Delta x)^2 + \Delta y$. This of course is really all that is being done, and probably Bob made this observation also.

The students were then asked to put their work on the board for problems (5), (6), (7).

For $y = x^2 - 8x + 27$, John wrote, "The vertex is (4,11) and it is the lowest point on the graph. I rearranged the formula to $y = (x + 4)^2 + 11$."

Allan said, "I got -11 when I did it."

John suggested writing down both answers and expanding them as a check. Thus:

$$y = (x + 4)^2 + 11 \qquad\qquad y = (x + 4)^2 - 11$$
$$= x^2 + 8x + 16 + 11 \qquad\qquad = x^2 + 8x + 16 - 11$$
$$= x^2 + 8x + 27 \qquad\qquad = x^2 + 8x + 5$$

It was evident then that John's answer was correct.

For $y = -x^2 - 12x - 3$ Pat wrote, "The vertex is (6, -39) and this is the highest point on the graph."

John said, "I got (6, -33)"

Allan said, "I got (-6, 33)."

Mike said, "I got (-6, -39)."

It was suggested that all these points be tested to see if they were on the graph, and it was found that (-6, 33) was the only point that satisfied the relation. Further checking assured the class that this was indeed the vertex.

Sharon was asked how she arrived at the vertex for $y = 2x^2 - 8x + 112$ to be the point (2,104).

"I took 2 out of the whole expression."

$$y = 2\left[x^2 - 4x + 56\right]$$
$$= 2\left[x^2 - 4x + 4 + 52\right]$$
$$= 2\left[(x - 2)^2 + 52\right]$$
$$= 2(x - 2)^2 + 104$$

At this point Keith raised the question, "How do you find the vertex of any one? Is there a formula you can use?"

Mike answered that the formula was "when $x = -b/2a$, the corresponding value of y is the greatest or the least."

The instructor hinted that Mike found the answer in the book, but in any case the exploratory period had been sufficient and the students were *not* harmed by this statement of the formula.

When further questioned on how this formula would be arrived at *Keith* answered, "I guess you would have to work out a question using letters instead of numbers."

The discussion terminated at this point, but one significant observation is pertinent:

The best discussion results when the students are given sufficient time to work and then the work is put on the board by the students. Variation in methods and exchanges of ideas on the part of the students make a good foundation for active classroom discussion. It would seem that the basis for the argument lies in the fact that a student is more apt to challenge the methods of another student, than he would a teacher imposed method.

VI ACTIVITY SIX

The sixth activity was one of summation on finding the vertex and plotting the quadratic function. However, in addition, the students were to find:

(1) axis of symmetry (defined in class)

(2) range (defined in class)

(3) x and y intercepts (defined in class)

The definitions given in class were not in terms of formulas, but rather in terms of what these concepts mean in relation to the graph.

Problems were assigned from text with the directions (a) plot (b) find (1), (2), (3) above.

(1) $y = x^2 - 6x + 8$	(5) $y = -x^2 + 2\sqrt{2}x - 3$
(2) $y = x^2 - 4x + 3$	(6) $y = -3x^2 - 2x$
(3) $y = 4x^2 - 4x - 15$	(7) $y = x^2 + 4x + 4$
(4) $y = x^2 - 10x + 9$	(8) $y = x^2 + 4x + 6$

The students had little previous knowledge of these concepts, and as they were working they were asked to look for possible formulas for finding:

(1) axis of symmetry

(2) range

(3) intercepts

We discussed the meaning of formula as being a short cut to getting answers.

Many pupils say "Why do we need formulas? Why not just work out the answers?" However, a few students in the class wanted short cut ways (formulas) for getting the answers.

Several students were working out the x-intercepts by getting the quadratic function in the form:

$$y = (x - k)^2 + 1, \text{ setting } y = 0$$

$$\text{i.e., } (x - k)^2 + 1 = 0$$

$$(x - k)^2 = -1 \qquad \text{etc.,}$$

This is of course the handiest way of solving the corresponding quadratic equation.

Other students were simply graphing the quadratic function and observing the points of intersection with the X-axis.

In any case the exercise seems to have served the purpose that most exploratory exercises must serve, namely, allowing the students to see that the "thing" exists by itself and that a formula is merely a means of finding it quickly. This set of problems, besides serving the function of a summation activity leads directly to the problem of finding the roots of quadratic equations and the character of these roots.

After allowing sufficient time for the students to examine these problems on their own, some of them were asked to put their work on the board and a discussion of methods followed.

(1) The y-intercept was easily observed to be the value of the absolute term in the expression.

(2) The range was associated with finding the vertex and especially the y-co-ordinate of the vertex.

(3) The axis of symmetry was arrived at through leading questions to be $x = -b/2a$, and the vertex was formed by associating this value of x with the maximum or minimum value of y.

(4) Various methods were employed in finding the x-intercepts and are worth while recording here:

(i) plotting points, drawing the graph, and then estimating the x-intercepts.

(ii) finding the vertex, using the formula, then using the format 1 over, 1 up, etc., for plotting the graph, and then estimating the x-intercepts.

(iii) setting $y = 0$ in the "standard" form, then factoring to find the x-intercepts.

(iv) putting the quadratic in the form, $y = (x + k)^2 + 1$, setting $y = 0$ and then solving for x.

Following this discussion, "finding the x-intercepts" was equated with "solving a quadratic equation." Another set of problems in which the students were asked to find the roots of the following quadratic equations were then assigned.

(1) $x^2 + 4x + 2 = 0$ (4) $4x^2 + 7x + 6 = 0$

(2) $(x + 2)^2 - 3 = 0$ (5) $4(x + 8)^2 + 6 = 0$

(3) $-4x^2 + 7x - 2 = 0$ (6) $x^2 + 2\sqrt{6}x + 6 = 0$

They were asked also to make a graph of the equation that had no roots. Problems arose when the graph did not cross the X-axis or when surd values occured, or when the graph was a tangent to the X-axis, as is the case in (6). In some cases they couldn't find exact roots; they were sure there weren't any; yet by plotting the graph they could see that there really were. This could be an example of "cognitive dissonance" (to use a term of Richard Suchman). They became really anxious at this point, and perhaps this is the closest yet to what is wanted in the Mathematizing Mode--a real problem sensed by the students--"How do we solve it?"

One question that arose from a discussion of these problems was "How do we tell a perfect square?" Several suggestions were offered in terms of the coefficients of $ax^2 + bx + c$:

(1) $(b/a)^2 = c$ (3) $b = 2ac$

(2) $a\sqrt{b} = c$ (4) $\sqrt{a} \times \sqrt{c} = b/2$

The students were asked to test these out on known trinomial squares and the class came to an agreement on the relationship

$b^2 = 4ac$ which is really another form of (4).

The question was then posed in a different form:

When is $(x + \square)^2 = \triangle$ a perfect square?

From here the class perceived that:

(1) when $\triangle = 0$, a perfect square

(2) when $\triangle$ is positive, real roots

(3) when $\triangle$ is negative, the graph will not cross the X-axis.

Actually the class had some idea that:

$$\triangle = \frac{b^2 - 4ac}{4a^2}$$

The previous problem was then restated.

If $b^2 = 4ac$ then the equation is a perfect square, meaning the roots are equal or the graph is a tangent to the X-axis.

(1) what are the conditions that roots be non real and the graph does not touch the X-axis?

(2) what are the conditions that the roots be real and not equal and the graph cuts the X-axis in two points?

It was Bryson who hypothesed:

(1) when $b^2 < 4ac$, the roots are non real

(2) when $b^2 > 4ac$, the roots are unequal and real.

The above hypotheses were then related to the form

$$(x - \square)^2 = \frac{b^2 - 4ac}{4a^2}$$, and through discussion of this form it became obvious that the above hypotheses were valid.

There was also some discussion about complex numbers. The students had no conception of why you could make them up. However, the discussion was not prolonged.

It must be observed that some of these results were difficult to get from the class and many students at this point were still not convinced of the results.

VII ACTIVITY SEVEN

The seventh activity was aimed at arriving at an expression (formula) for the roots of $ax^2 + bx + c = 0$. Once found, the students were asked to be prepared to prove their results. The instructor chose to take the general form $ax^2 + bx + c = 0$ down to the completed square form

$$(x + b/2a)^2 = \frac{b^2 - 4ac}{4a^2}$$

We discussed again briefly the possibilities of roots being real, non real, or equal and the problem was stated:

Given $ax^2 + bx + c = 0$ (1) What is the formula for finding the roots?
(2) How would you prove to someone that it is true?

Again many of the students said "We can solve the equation--why do we want a formula?" They were not really motivated to find formulas.

After a few minutes, Allan yelled "I've got it" but other than that there were really no great insights. Only about half of the students had the formula before the instructor began to develop it.

$$(x + b/2a)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + b/2a = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

After discussion with the class, everyone was quite clear on its derivation.

The students were then assigned problems from the text.

1. Determine the character of the roots of each of the following:

   (a) $12x^2 - x - 6 = 0$ (d) $4y^2 + 7y + 4 = 0$

   (b) $3y^2 - 4y + 10 = 0$ (e) $\sqrt{5}z^2 - 3z - \sqrt{5} = 0$

   (c) $z^2 - 8z + 16 = 0$ (f) $\sqrt{3}x^2 - 2x - \sqrt{3} = 0$

2. Determine the real values of k for which each of the following equations has real and equal roots:

   (a) $3x^2 - 4x - (3 + k) = 0$

   (b) $x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$

3. For what values of k does the equation 2(b) have:

   (a) real and unequal roots

   (b) non-real roots.

They were also asked to find roots and plot the graphs in the case of non-real roots. This served as a built-in check. A good discussion of all of these problems followed, and we feel that the principles involved were grasped by most of the students.

VIII ACTIVITY EIGHT

The eigth activity was a reversal of form from the previous one. Roots of equations were given and the students were asked to find the equations. This is more along the lines of a "constructive" mode of working as opposed to the "analytic" approach taken previously. It really served an introductory activity directed at arriving at formulas for the sum and product of the roots.

Discussion centered about ways of finding equations, given the roots. The following problems were then assigned.

Form the equation whose roots are:

(1) 1, 2	(4) 7/9, -9
(2) $\frac{1}{2}$, 4	(5) $\sqrt{2}$, $\sqrt{3}$
(3) 3/2, -2	(6) p, q

After allowing the students to work out the above problems, they were asked to formulate a rule for finding:

(1) the sum of the roots

(2) the product of the roots

When the co-efficient of x^2 is one, they had little trouble generalizing that the sum of the roots was -b and that the product of the roots was c.

After further discussion we ended up with the generalization:

sum of the roots = -b/a

product of the roots = c/a

A series of problems were then assigned.

1. Find without solving the equation, the sum and product of the roots of each of the following quadratic equations.

(a) $4z^2 + 28z + 53 = 0$

(b) $y^2 + 2y - 1 = 0$

(c) $2x^2 + x - 2 = 0$

2. (a) Show that not both 1/3 and -2 can be roots of the equation $3x^2 + 2x - 1 = 0$

(b) Given that 1/3 is a root, find the other root.

3. If the roots of $ax^2 - 11x + 2 = 0$ are reciprocals, find a.

4. If each of $7z^2 + bz + 5 = 0$ is the negative of the other, determine b.

5. If one root of the equation $x^2 + bx - 5 = 0$ is 1/2, determine b.

6. If one root of the equation $4z^2 - 12z + c = 0$ is twice the other root, determine c.

7. Determine the condition that the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ has

(1) reciprocal roots

(2) roots which are the negative of each other

(3) one root zero.

A thorough discussion of the various procedures employed by the students in solving these problems followed. Most of the procedures were quite standard, but Keith offered something original and is worth noting.

For 5. Given that $\frac{1}{2}$ is one root of $x^2 + bx - 5 = 0$, determine b.

He suggested direct substitution, i.e.,

$$(\tfrac{1}{2})^2 + b\,(\tfrac{1}{2}) - 5 = 0$$

$$\tfrac{1}{4} + \tfrac{1}{2}b - 5 = 0$$

$$b/2 = 4\ 3/4$$

$$b = 9\tfrac{1}{2}$$

The class was now, we thought, ready to suggest procedures for proving the formulas

(a) Sum of roots $= -b/a$

(b) Product of Roots $= c/a$

Until this time they had only guessed at the formulas.

Two methods were used. The first was introduced completely by a student.

Method One. Let k, m be the roots

$$(x - k)(x - m) = 0 \quad \text{Then}$$

$$x^2 - (k + m)x + km = 0$$

sum product.

$$\text{i.e.} \quad x^2 - \frac{b}{a}x + \frac{c}{a} = 0$$

The other method took a little more time. In fact, even after the teacher wrote the two roots on the board as follows:

$$k = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Only a few students discovered the standard proof of

(1) $k + m = -b/a$

(2) $km = c/a$

IX ACTIVITY NINE

The ninth activity centered around the solution of maximum and minimum problems. Previous activities have prepared the students to accept this type of problem as an application of what has already been learned. The authors feel that in this case the problem is really one of "assimilating" a related topic into an already existing "cognitive structure." This is quite significant, and illustrates that if the original problems or activities of a unit are broad enough in scope, they will

encompass much of the knowledge to be learned within a given unit. This makes the problem of teaching individual problems within a unit much simpler.

There are really five kinds of questions supplied in the text as examples and exercises.

(1) area as a function of side

(2) height as a function of time

(3) product of two numbers as a function of the first

(4) total income at an event as a function of the price

(5) maximum-sum-given-product problems.

Clearly the central idea throughout this section is the idea of a function, and once again, the graphical approach proved to be beneficial.

Two problems of the maximum-area-given-perimeter type were outlined to the students and they were asked to solve these by means of a graph.

(1) Length of fence (total perimeter) is 1000 feet. Find maximum area.

(2) Length of fence (three sides) is 1000 feet. Find maximum area.

A brief initial discussion on the idea of area being a function of the side preceded the exploratory period.

e.g., if the length is 100 feet, area is 40,000 square feet.

if the length is 200 feet, area is 60,000 square feet.

Once the students were aware of the changing area with respect to the side, they were asked to solve the above two problems by means of drawing a graph.

Most of the students were able to isolate the relevant variables and to solve the problems. Here again the vertex-form of the quadratic function proved invaluable because the location of the vertex identifies the maximum or minimum value.

We would emphasize here the usefulness of the graphical approach as being the most descriptive to the above problem.

It is appropriate here that either the teacher or the pupils should initiate a discussion on the two aspects of these problems:

(1) application - i.e., formulation of a function--a mathematical description of the problem.

(2) Solution--i.e., using their background as a tool with which to solve a physical problem described in mathematical terms.

Following these problems and discussion, the students were assigned several questions from the text to test their

understanding of the activity.

Significantly, however, the teacher especially emphasized that he would not solve these problems for the students. He would only suggest possible hints, but the students were responsible for finishing all of the questions. This is important, because only too often the "role perception" of the teacher from the student's seat is one of "showing how." This "mind set" on the part of pupils must be broken down whenever possible.

We do feel that there are exceptions to the above, as in the case of "completed square" problems of the type $y = (\sqrt{x} + 1/\sqrt{x})^2$. These and other "mathematical tricks" may be shown to the students.

In all their work throughout this activity the students were reminded of the central idea--the function. They had to represent one aspect of the problem as a function of another in order to solve the question. They were also encouraged to illustrate their work graphically as an aid to understanding.

It is to be noted also that at this stage in the activity several students were using the formulas:

$x = -b/2a$, $y = 4ac - b^2/4a$, in order to find the vertex. Most often though, they were using them as a check, after finding the answer. One of the students

remarked that he was not checking his answer, but rather checking the validity of the formulas.

X. ACTIVITY TEN

The tenth activity centered around the solution of equations that can be handled like quadratics after an initial observation or re-arrangement. Here, as in the previous activity there are the two aspects of the problem to contend with--application and solution. The following questions were given to the students as an exercise and following a brief discussion on the actual statement of the problem, they were set to work on finding solutions:

(1) $\frac{2x + (1)}{x - 3} - \frac{x + 4}{3} = \frac{8}{5}$

(2) $x^4 - 4x^2 - 12 = 0$

(3) $\frac{x + 3}{x} \quad 5 = x + \frac{3}{x}$

(4) $(x^2 + 3x - 1)(x^2 + 3x - 2) = 6$

(5) $x = 5 + \quad 1 + x$

(6) $\sqrt{4x - 3} = 1 + \sqrt{x + 1}$

(7) $\sqrt{4 - 3w} - w = 12$

After a brief statement on what it meant to solve an equation, the exploratory period of 15-20 minutes began. A discussion was then initiated which soon revealed that major difficulties were evident in the solution of 2 and 4 above.

It was suggested that the formula be used to attack (1). It resembled a quadratic where $a = x^2$, $b = -4x$, $c = -12$. The class did not pursue this suggestion, although it does work; instead they preferred another suggestion of factoring the expression.

$$(x^2 - 6)(x^2 + 2) = 0$$

$$x = \pm\sqrt{6}, \quad x = \pm\sqrt{-2}$$

At this point the instructor attempted to answer the question "How can we make it look like a quadratic equation?" Again, after some discussion he suggested that if we let $x^2 = \square$, the equation would be . . . The students had little trouble finding the statement:

$$\square^2 - 4\square - 12 = 0$$

Here it was suggested that we use m instead of $\square$ and the equation was rewritten.

$$m^2 - 4m - 12 = 0$$

It was left at this point for the students to finish.

Upon examining (6) a substitution was suggested; $m = x^2 + 3x$ was chosen. Here again the students were left to finish by themselves.

The students were then told that they had to finish the questions by themselves and be prepared to outline their method of solution.

It was quite evident from observation during their exploratory period that many students were now using the

quadratic formula in favor of other methods discussed. In fact, the students find that once the numbers get large and unwieldy, it is imperative to use the quadratic formula.

It is suggested that at this point a good summing up exercise would be to take a quadratic equation and solve it using all four methods:

(1) factoring

(2) completing the square

(3) using the formula

(4) graphing the corresponding function.

XI ACTIVITY ELEVEN

A "summing up quadratics" handout was prepared[2] and this paper was used as the basis of discussion for one or two periods. The procedure was quite straightforward and the general ideas were to bring all their explorations and discoveries into perspective as well as give them formulas to memorize.

Special attention was given to the meaning of the following terms:

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[2] *cf.* *post* p. 112, Summing Up Quadratics.

(1) function, (2) equation, (3) domain, (4) range, (5) solve, (6) roots.

The discussion on functions and equations led to the consideration of such expressions as

(1) $y = x^2 + 2$

(2) $y = 0$

(3) $4x + 2 = 0$

(4) $x + y + 4 = 0$; $y = -x - 4$; $x + y = -4$

(5) $y + 3 = 0$

We concluded that the difference between the two names was associated with the number of variables involved. For example $x + 4 = 0$ is an equation, but put in set-builder notation $\{(x, y) \mid x + 4 = 0\}$ is a relation and not simply an equation.

It is appropriate that some time be devoted to a discussion on this topic because it is a source of much confusion to the pupils.

It was particularly interesting and encouraging to note the number of questions the students had which explored other possibilities. They did however show concern for getting through the summing up activity by "ruling out" some of the questions from discussion.

Perhaps the central theme of the entire activity is that all we really did was put names and formulas on things which they had explored freely during the exploratory

sessions.

In summing up types of problems, examples from the book were used to illustrate various approaches to any given problem. It was observed that all problems do not fall into a given type, but some demand an original solution. It was also observed that some problems are simplified by the use of a "mathematical trick" or an astute observation. In actuality this type of review and summation invariably ends up in "drill" fashion. It is to be hoped however, that the type of originality shown when the student first encounters a type of problem will not be stifled by a recognition of the standard approach involved.

SUMMING UP QUADRATICS

1. $y = ax^2 + bx + c$ - a general quadratic function
2. $y = 2x^2 + 7x - 12$ - an example of a special quadratic function
3. $y = 2(x - \frac{5}{2})^2 - 21$ - an example of a quadratic function in the completed square form.
   - where $\frac{5}{2}$ is the x coordinate of the vertex
   - where -21 is the y coordinate of the vertex
4. $\frac{-b}{2a}$ - x coordinate of the vertex

5. $4ac - b^2$, the y coordinate of the vertex.

6. range of the function, the values of y coordinate.

7. $x = \frac{-b}{2a}$, the axis of symmetry

8. maximum and minimum values (vertex)

9. intercepts of the function

10. solution of quadratic equations:

   (a) $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

   $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

   (b) factoring

   (c) completing the square

   (d) graph the function

11. sum of the roots $= \frac{-b}{a}$

12. product of roots $= + \frac{c}{a}$

13. characteristics of roots:

   (a) $b^2 - 4ac = 0$ two equal roots (tangent to the X-axis).

   (b) $b^2 - 4ac > 0$ two real, unequal roots (cuts the X-axis).

   (c) $b^2 - 4ac < 0$ two non-real, unequal roots (does not cut the X-axis).

14. Types of problems:

   (a) Formation of quadratic equations given the roots.

   (b) Properties of roots determined by the coefficients of the quadratic equation.

(c) Maximum and minimum problems.

(d) Equations solved like quadratics.

(e) Graphing function to illustrate vertex, intercepts, axis of symmetry, domain and range.

SUMMARY

The same general pattern of development is observed in chapter five as was illustrated in the previous chapter. Although the actual stages of instruction and their accompanying behaviors are not identified, it is felt that they are, in fact, obvious to the reader. The proceedings of this chapter indicate that the Mathematizing Mode can be applied to the unit involving the quadratic function and related topics. This unit is generally introduced at the grade eleven level.

CHAPTER VI

CONCLUSIONS

The outline of the final chapter of this thesis is as follows:

(1) A brief summary of the problem under investigation.

(2) A look in retrospect at chapters two, three, four and five in order to better clarify the contribution of each to the solution of the problem, and also to indicate clearly the results of this study.

(3) The implications that this study has for further research in education.

The first two topics are grouped under the heading, "Summary and Results." The third topic is treated separately under the heading, "Implications for Further Research."

Because of the nature of the present study, results are reported on an observational level. There is no attempt to apply mathematical statistics to an analysis of the effectiveness of discovery teaching. The purpose of the study is one of description, hence, the results of the study centre around the nature and applicability of this description.

I. SUMMARY AND RESULTS

The problem presently under investigation is concerned with a current dilemma persisting among researchers in mathematics curriculum. The two aspects of curriculum are content and method. Much energy and expense have been spent in developing and organizing new content. At the same time, these researchers express an almost unanimous recommendation that the new content be presented in the atmosphere of discovery. Herein lies the dilemma. Little attempt has been made either to clearly define this discovery mode of instruction or, once defined, to illustrate its use in a specific instance. It is the objective of this study to present a description and illustration of one such discovery mode, referred to in this thesis as the Mathematizing Mode. The approach to the problem is seen in the organization and sequencing of chapters two, three, four and five.

Chapter two

This chapter contributes only in a general way to a description of the Mathematizing Mode. The theoretical position of each author reviewed is examined and the common characteristics of discovery teaching are isolated. As a result of this review it is concluded:

(1) Most authors reviewed agree that there exists much confusion as to the actual process of discovery teaching.

(2) Several characteristics of discovery teaching emerge as common.

(3) There is some divergence of thought regarding the role of discovery teaching in the classroom. Should it be the primary means of instruction or should it be used in a supplemental fashion?

The results of chapter two are used as a basis for the development of the theoretical framework of the Mathematizing Mode.

Chapter Three

The theoretical framework of the Mathematizing Mode is presented in this chapter. Here, the theoretical positions examined above are integrated into a concise description of the Mathematizing Mode. Four stages of instruction are recommended: (1) problem situation, (2) accepting hypotheses from students, (3) evaluation of these hypotheses and, (4) a precise mathematical description of the learning experiences. In order to distinguish the Mathematizing Mode from other forms of discovery teaching and from other modes of instruction, specific teacher and pupil behaviors are prescribed in reference to each of the above mentioned four stages.

Reference will be made to these behavior patterns in the discussion of chapters four and five.

Two results of chapter three are clearly evident:

(1) The Mathematizing Mode is a form of discovery teaching.

(2) The Mathematizing Mode is well described in that the theoretical framework is reduced to behavioral terms for the teacher and pupils.

The above description of the Mathematizing Mode is now applied to developing and teaching specific content from the senior high school curriculum.

Chapter Four and Five

An illustration of the use of the Mathematizing Mode as applied to developing and teaching two units of instruction is shown in chapters four and five. A reference is now made to the teacher and pupil behaviors prescribed in the theoretical framework. For each behavior pattern, a specific example is chosen from the foregoing outline of "The Linear Function," and "The Quadratic Function." This approach is necessary for two reasons:

(1) To assure the reader that the theoretical framework described in chapter three is indeed the one that is illustrated in chapters four and five.

(2) To assess the feasibility of using the Mathematizing Mode to develop and teach mathematics in the senior high school.

Teacher Behaviors

(1) The teacher must wait for the students to verbalize the statement of the problem.

During the activity on plotting the graph of a quadratic, the students recognized the importance of finding the vertex. This served as an activity for a new stage one.[1]

(2) If the students do not pose a significant problem, the teacher must supply the problem within a broad context to allow at least for student participation in exact formulation of the problem.

In the linear function such problems are found as, "give the name for this line," or "find the distance between two points."[2]

(3) The teacher should accept different problems arising from the same situation as in the brainstorming session of stage two. Every suggestion must be accepted and recorded.

Several methods are used during the course

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[1] *cf.* *ante* pp. 84,88. [2] *cf.* *ante* pp. 64,76.

of the two units, such as writing down the hypotheses on the board as they are presented or copying them down in a notebook, or immediately projecting the students hypotheses with the overhead.[3]

(4) The teacher should accept wrong or partially correct hypotheses with equal enthusiasm.

"The sum of the co-ordinates of points along a special diagonal are the same." "Lines are perpendicular if their slopes are negative of each other."[4]

(5) The teacher must convey to the students that unless they participate in an active, energetic fashion, there will be no further class and consequently no learning.

Sufficient time is allowed during the exploratory period before hypotheses are accepted from the class.[5]

(6) The teacher should encourage the use of intermediate language.

Examples of this are: "special diagonal and ordinary diagonal," or "tip of the parabola."[6]

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[3] cf. ante pp. 66,70,84.

[4] cf. ante pp. 67,73.

[5] cf. ante pp. 66,83,85,95.

[6] cf. ante pp. 68,88.

(7) The teacher must be willing to accept several methods of solution to the problem.

Two examples are: "Two methods of approaching the slope-point form of a line," "Four suggested procedures for find the x-intercepts of a quadratic."[7]

(8) In addition to stating problems, the teacher will have to state solutions to problems. This latter especially refers to definitions and conventions.

The definition of the slope of a line was given as $\frac{\Delta y}{\Delta x}$.[8]

(9) The teacher should not hesitate to assign problems before the students know formulas for solving them. If the students have a "mind-set" that they should always be looking for short-cuts, then they will seek them out as they solve these problems. The main idea here is that the students explore the problem situation independently, or in groups of three or four before any class discussion is held.

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[7] *cf.* *ante* pp. 74,96. [8] *cf.* *ante* p. 71.

Two examples are: "Find a rule for a line on two points," "Find the highest or lowest points of the graphs of the following quadratic functions."[9]

(10) The teacher should be prepared to leave an activity before it is completely resolved, and proceed with another. In addition, he should not become anxious if some students develop inexact or incomplete ideas.

Activity four of the quadratic function left much to be desired in finding the completed square form of the quadratic. It was abandoned at this point, and during activity five was resolved from a different direction.[10]

Pupil Behaviors

(1) In their individual work habits, the students should exhibit a "mind set" that mathematics means looking for best ways, short-cuts, patterns, or generalizations in the solution of problems.

An example of this would be: "Slope of line is $-A/B$ in $Ax + By + C = 0$."[11]

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[9] cf. ante pp. 72,75,90.

[10] cf. ante p. 87.

[11] cf. ante pp. 73,75,84,94.

(2) A willingness to contribute answers, basically thinking out loud during class discussion. Their hypotheses will not at first be well formulated, but are presented so that other students will get the benefit of their first-order thinking.

One example is predicting points on the graph of $y = x^2$ as 1 over 1 up, 2 over 4 up etc. from the vertex.[12]

(3) The student should not only be continually looking for solutions to the problem at hand, but also stating new and significant problems.

While plotting the graphs of several quadratic functions, the students realized the necessity of knowing the position of the vertex--a new and significant problem.[13]

(4) The student should be continually evaluating and testing hypotheses, expecting that many of the statements made in class by other students are wrong or only partially correct. At this point, they should be quite adept in using a counter example to disprove a hypothesis. They should also be aware that many of the hypotheses are really the same one, in different words. This latter will show them the benefits of stating

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[12] cf. ante pp. 81,82. [13] cf. ante pp. 84,88.

it in symbolic, conventional language.

A review of the second activity in the unit on quadratics illustrates this desirable behavior on the part of the pupil.[14]

(5) The student should be engaged in the use of formulas and symbolism in the solution of problems.

Two examples of this would be the use of the quadratic formula for finding the roots of a given equation and the use of the distance formula for finding the distance between points.[15]

The description of the theoretical framework in terms of stages of instruction, and the further description of each stage in terms of teacher pupil behavior help to make the Mathematizing Mode a distinctive mode of instruction.

It is concluded from a review of the proceedings outlined in chapters four and five that:

(1) The theoretical framework illustrated in the development and teaching of "The Linear Function," and "The Quadratic Function" is indeed the theoretical framework described in chapter three.

(2) The Mathematizing Mode can be used to develop and teach at least these two units of mathematical content.

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[14]<u>cf</u>. <u>ante</u> pp. 82-85. [15]<u>cf</u>. <u>ante</u> pp. 72,77.

It is further concluded that it is feasible to use the Mathematizing Mode to develop and teach other topics from the senior high school curriculum. This conclusion is a possible intrusion on the implications of this study, but the author feels that this extrapolation of results is justified.

II. IMPLICATIONS FOR FURTHER RESEARCH

Several areas of research are suggested by the present study. One suggestion is that an evaluation of the effectiveness of the Mathematizing Mode is necessary before attempting to implement it within the curriculum. This aspect of evaluation is difficult to assess due to the deficiency of adequately prepared teachers. An alternate approach would be to conduct research into the re-training of teachers through inservice courses. It is hardly fair to ask teachers who have received training in the expository mode, and perhaps have employed this method for several years, to perform a complete turn about for the purpose of evaluating the Mathematizing Mode.

Results of this study indicate that it is feasible to teach at least two topics in the Mathematizing Mode. Further research could develop other topics of the grade eleven curriculum in the Mathematizing Mode. An outline

of this type would be most valuable to teachers who are interested in discovery teaching, but who have difficulty reducing it to a well defined process of instruction. After a suitable trial period, evaluation would then seem more appropriate.

For purposes of comparative studies, relating to evaluation of teaching methods, a similar description of an expository mode or of Ausubel's process of "reception learning" would be extremely beneficial.

Most important of all, however, is a thorough investigation of Bruner's hypothesis. Does learning by the discovery mode really develop an ability in the student to make further discoveries more easily and more rapidly? It is only an answer to this that will break down the time-honored conviction that learning by discovery is too time comsuming. A breakthrough in this area would convince both administrators and teachers of the true value of a discovery mode of instruction, and open the way for much research of the types suggested.

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